

# FOUR-VARIABLE $p$ -ADIC TRIPLE PRODUCT $L$ -FUNCTIONS AND THE TRIVIAL ZERO CONJECTURE

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ABSTRACT. We construct the four-variable primitive  $p$ -adic  $L$ -functions associated with the triple product of Hida families and prove the explicit interpolation formulae at all critical values in the balanced range. Our construction is to carry out the  $p$ -adic interpolation of Garrett's integral representation of triple product  $L$ -functions via the  $p$ -adic Rankin-Selberg convolution method. As an application, we obtain the cyclotomic  $p$ -adic  $L$ -function for the motive associated with the triple product of elliptic curves and prove the trivial zero conjecture for this motive.

## CONTENTS

|    |  |    |
|----|--|----|
| 1. | Introduction   | 1  |
| 2. | Computation of the local zeta integral: the $p$ -adic case   | 6  |
| 3. | Computation of the local zeta integral: the ramified case    | 14 |
| 4. | Computation of the local zeta integral: the archimedean case | 17 |
| 5. | Classical and $p$ -adic modular forms                        | 25 |
| 6. | A $p$ -adic family of pull-backs of Siegel Eisenstein series | 28 |
| 7. | Four-variable $p$ -adic triple product $L$ -functions        | 33 |
| 8. | The trivial zero for the triple product of elliptic curves   | 39 |
|    | References   | 46 |

## 1. INTRODUCTION

The aim of this paper is to construct the four-variable  $p$ -adic triple product  $L$ -functions for the triple product of Hida families of elliptic newforms with explicit interpolation formulae at all critical specializations in the balanced region. Let  $p$  be an *odd* prime,  $\mathcal{O}$  a valuation ring finite flat over  $\mathbf{Z}_p$  and  $\mathbf{I}$  a normal domain finite flat over the Iwasawa algebra  $\Lambda = \mathcal{O}[[\Gamma]]$  of the topological group  $\Gamma = 1 + p\mathbf{Z}_p$ . Let

$$\mathbf{F} = (\mathbf{f}, \mathbf{g}, \mathbf{h})$$

be a triplet of primitive Hida families of tame conductor  $(N_1, N_2, N_3)$  and nebentypus  $(\chi_1, \chi_2, \chi_3)$  with coefficients in  $\mathbf{I}$ . Roughly speaking, we construct a four-variable Iwasawa function that interpolates the algebraic part of critical values of the triple product  $L$ -function attached to  $\mathbf{F}$  at all balanced critical specializations twisted by Dirichlet characters. Our formulae completely comply with the conjectural form described in [CPR89], [Coa89a] and [Coa89b]. In order to state our result precisely, we need to introduce some notation from Hida theory for elliptic modular forms and technical items such as the modified Euler factors at  $p$  and the canonical periods of Hida families in the theory of  $p$ -adic  $L$ -functions.

**1.1. Galois representations attached to Hida families.** Given a field  $F$ , we denote its separable closure by  $\overline{F}$  and put  $G_F = \text{Gal}(\overline{F}/F)$ . If  $\mathcal{F} = \sum_{n=1}^{\infty} \mathbf{a}(n, \mathcal{F})q^n \in \mathbf{I}[[q]]$  is a primitive cuspidal Hida family of tame conductor  $N_{\mathcal{F}}$  and nebentypus  $\chi_{\mathcal{F}}$ , let  $\rho_{\mathcal{F}} : G_{\mathbf{Q}} \rightarrow \text{GL}_2(\text{Frac } \mathbf{I})$  be the associated big Galois representation such that  $\text{Tr } \rho_{\mathcal{F}}(\text{Frob}_{\ell}) = \mathbf{a}(\ell, \mathcal{F})$  for primes  $\ell \nmid N_{\mathcal{F}}$ , where  $\text{Frob}_{\ell}$  is the geometric Frobenius at  $\ell$  and  $V_{\mathcal{F}}$  is the natural realization of  $\rho_{\mathcal{F}}$  inside the étale cohomology groups of modular curves. Thus  $V_{\mathcal{F}}$  is a lattice in

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*Date:* July 23, 2019.

2010 *Mathematics Subject Classification.* 11F67, 11F33.

Hsieh was partially supported by a MOST grant 108-2628-M-001-009-MY4. Yamana is partially supported by JSPS Grant-in-Aid for Scientific Research (C) 18K03210.

$(\text{Frac } \mathbf{I})^2$  with the continuous Galois action via  $\rho_{\mathcal{F}}$ , and the  $G_{\mathbf{Q}_p}$ -invariant subspace  $\text{Fil}^0 V_{\mathcal{F}} := V_{\mathcal{F}}^{I_p}$  fixed by the inertia group  $I_p$  at  $p$  is free of rank one over  $\mathbf{I}$  ([Oht00, Corollary, page 558]). A point  $Q \in \text{Spec } \mathbf{I}(\overline{\mathbf{Q}}_p)$  is called an arithmetic point if  $Q|_{\Gamma} : \Gamma \hookrightarrow \Lambda^\times \xrightarrow{Q} \overline{\mathbf{Q}}_p^\times$  is given by  $Q(x) = x^{k_Q} \epsilon_Q(x)$  for some integer  $k_Q \geq 2$  and a finite order character  $\epsilon_Q : \Gamma \rightarrow \overline{\mathbf{Q}}_p^\times$ . Let  $\mathfrak{X}_{\mathbf{I}}^+$  be the set of arithmetic points of  $\mathbf{I}$ . For each arithmetic point  $Q \in \mathfrak{X}_{\mathbf{I}}^+$ , the specialization  $V_{\mathcal{F}_Q} := V_{\mathcal{F}} \otimes_{\mathbf{I}, Q} \overline{\mathbf{Q}}_p$  is the geometric  $p$ -adic Galois representation associated with the  $p$ -stabilized newform  $\mathcal{F}_Q = \sum_{n=1}^{\infty} Q(\mathbf{a}(n, \mathcal{F}))q^n$ .

**1.2. Triple product  $L$ -functions.** We denote by  $\mathbf{Q}_{\infty}$  the cyclotomic  $\mathbf{Z}_p$ -extension of  $\mathbf{Q}$ , by  $\omega : G_{\mathbf{Q}} \rightarrow \mu_{p-1} \hookrightarrow \mathbf{Z}_p^\times$  the Teichmüller character, by  $\varepsilon_{\text{cyc}} : \text{Gal}(\mathbf{Q}_{\infty}/\mathbf{Q}) \xrightarrow{\sim} 1 + p\mathbf{Z}_p = \Gamma$  the  $p$ -adic cyclotomic character and by  $\langle \varepsilon_{\text{cyc}} \rangle_T : G_{\mathbf{Q}} \rightarrow \text{Gal}(\mathbf{Q}_{\infty}/\mathbf{Q}) \hookrightarrow \mathbf{Z}_p[[\text{Gal}(\mathbf{Q}_{\infty}/\mathbf{Q})]]^\times$  the universal cyclotomic character. Let

$$\mathbf{I}_3 = \mathbf{I} \widehat{\otimes}_{\mathcal{O}} \mathbf{I} \widehat{\otimes}_{\mathcal{O}} \mathbf{I}, \quad \mathbf{I}_4 = \mathbf{I}_3[[\text{Gal}(\mathbf{Q}_{\infty}/\mathbf{Q})]]$$

be finite extensions of the three and four-variable Iwasawa algebras.

Fix  $a \in \mathbf{Z}/(p-1)\mathbf{Z}$ . The main object of this paper is a construction of the  $p$ -adic  $L$ -function for the triple tensor product Galois representation

$$\mathcal{V} = V_{\mathbf{f}} \widehat{\otimes}_{\mathcal{O}} V_{\mathbf{g}} \widehat{\otimes}_{\mathcal{O}} V_{\mathbf{h}}, \quad \mathbf{V} = \mathcal{V} \widehat{\otimes}_{\mathcal{O}} \omega^a \langle \varepsilon_{\text{cyc}} \rangle_T$$

of rank eight over  $\mathbf{I}_4$ . If  $(k_1, k_2, k_3)$  is a triplet of positive integers, we say  $(k_1, k_2, k_3)$  is balanced if  $k_1 + k_2 + k_3 > 2k^*$  with  $k^* := \max\{k_1, k_2, k_3\}$ . Let  $\mathfrak{X}_{\mathbf{I}_3}^{\text{bal}}$  denote the set of balanced arithmetic points of  $(\mathfrak{X}_{\mathbf{I}}^+)^3$ . An integer  $k$  is said to be critical for  $(k_1, k_2, k_3)$  if

$$k^* \leq k \leq k_1 + k_2 + k_3 - k^* - 2.$$

We define the weight space  $\mathfrak{X}_{\mathbf{I}_4}^{\text{bal}} \subset \text{Spec } \mathbf{I}_4(\overline{\mathbf{Q}}_p)$  to be the set of balanced critical points of  $\mathbf{I}_4$  given by

$$\mathfrak{X}_{\mathbf{I}_4}^{\text{bal}} = \{(Q_1, Q_2, Q_3, P) \in \mathfrak{X}_{\mathbf{I}_3}^{\text{bal}} \times \mathfrak{X}_{\Lambda}^+ \mid k_P \text{ is critical for } (k_{Q_1}, k_{Q_2}, k_{Q_3})\}.$$

For each point  $(Q, P) = (Q_1, Q_2, Q_3, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$ , the specialization  $\mathbf{V}_{(Q, P)} = \mathcal{V}_Q \otimes \varepsilon_{\text{cyc}}^{k_P} \epsilon_P \omega^{a-k_P}$  is a  $p$ -adic geometric Galois representation, where  $\mathcal{V}_Q = V_{\mathbf{f}_{Q_1}} \otimes V_{\mathbf{g}_{Q_2}} \otimes V_{\mathbf{h}_{Q_3}}$  and  $\epsilon_P$  is regarded as a Galois character via  $\epsilon_P \circ \varepsilon_{\text{cyc}}$ .

Next we briefly recall the motivic  $L$ -function associated with the specialization  $\mathbf{V}_{(Q, P)}$ . To the geometric  $p$ -adic Galois representation  $\mathbf{V}_{(Q, P)}$ , we can associate the Weil-Deligne representation  $\text{WD}_{\ell}(\mathbf{V}_{(Q, P)})$  of the Weil-Deligne group of  $\mathbf{Q}_{\ell}$  over  $\overline{\mathbf{Q}}_p$  (See [Tat79, (4.2.1)] for  $\ell \neq p$  and [Fon94, (4.2.3)] for  $\ell = p$ ). Fixing an isomorphism  $\iota_p : \overline{\mathbf{Q}}_p \simeq \mathbf{C}$  once and for all, we define the motive  $L$ -function of  $\mathbf{V}_{(Q, P)}$  by the Euler product

$$L(\mathbf{V}_{(Q, P)}, s) = \prod_{\ell < \infty} L_{\ell}(\mathbf{V}_{(Q, P)}, s)$$

of the local  $L$ -factors  $L_{\ell}(\mathbf{V}_{(Q, P)}, s)$  attached to  $\text{WD}_{\ell}(\mathbf{V}_{(Q, P)}) \otimes_{\overline{\mathbf{Q}}_p, \iota_p} \mathbf{C}$  (cf. [Del79, (1.2.2)], [Tay04, page 85]). On the other hand, we denote by  $\pi_{\mathbf{f}_{Q_1}}$  (resp.  $\pi_{\mathbf{g}_{Q_2}}, \pi_{\mathbf{h}_{Q_3}}$ ) the irreducible unitary cuspidal automorphic representation of  $\text{GL}_2(\mathbf{A})$  associated with  $\mathbf{f}_{Q_1}$  (resp.  $\mathbf{g}_{Q_2}, \mathbf{h}_{Q_3}$ ). Let  $L(s, \pi_{\mathbf{f}_{Q_1}} \times \pi_{\mathbf{g}_{Q_2}} \times \pi_{\mathbf{h}_{Q_3}} \otimes \epsilon_P \omega^{a-k_P})$  be the automorphic  $L$ -function attached to the triple product of  $\pi_{\mathbf{f}_{Q_1}}$ ,  $\pi_{\mathbf{g}_{Q_2}}$ , and  $\pi_{\mathbf{h}_{Q_3}} \otimes \epsilon_P \omega^{a-k_P}$ , as constructed by Garrett [Gar87] in the classical setting and by Piatetski-Shapiro and Rallis [PSR87] in the adèlic setting. The analytic theory of  $L(s, \pi_{\mathbf{f}_{Q_1}} \times \pi_{\mathbf{g}_{Q_2}} \times \pi_{\mathbf{h}_{Q_3}} \otimes \epsilon_P \omega^{a-k_P})$  such as meromorphic continuation and a functional equation has been explored extensively in the literatures (cf. [PSR87, Ike89, Ike92]), and thanks to [Ram00, Theorem 4.4.1], we have

$$L(s + k_P - w_Q/2, \pi_{\mathbf{f}_{Q_1}} \times \pi_{\mathbf{g}_{Q_2}} \times \pi_{\mathbf{h}_{Q_3}} \otimes \epsilon_P \omega^{a-k_P}) = \Gamma_{\mathbf{V}_{(Q, P)}}(s) \cdot L(\mathbf{V}_{(Q, P)}, s),$$

where  $w_Q := k_{Q_1} + k_{Q_2} + k_{Q_3} - 3$  and  $\Gamma_{\mathbf{V}_{(Q, P)}}(s)$  is the Gamma factor of  $\mathbf{V}_{(Q, P)}$  as given by

$$\Gamma_{\mathbf{V}_{(Q, P)}}(s) := \Gamma_{\mathbf{C}}(s + k_P) \Gamma_{\mathbf{C}}(s + 1 + k_P - k_{Q_1}) \Gamma_{\mathbf{C}}(s + 1 + k_P - k_{Q_2}) \Gamma_{\mathbf{C}}(s + 1 + k_P - k_{Q_3}).$$

Here  $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$ . Hence we have a good understanding of the analytic properties of the motivic  $L$ -function  $L(\mathbf{V}_{(Q, P)}, s)$ . The rationality of its critical  $L$ -values in the balanced region was proved in [Orl87] and [GH93], where the authors verify that the Deligne's period for  $\mathbf{V}_{(Q, P)}$  is the product of Petersson norms of  $\mathbf{f}_{Q_1}$ ,

$g_{Q_2}, h_{Q_3}$ . In this article we shall investigate the arithmetic of critical values  $L(\mathbf{V}_{(\underline{Q}, P)}, 0)$  for  $(\underline{Q}, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$  and study the  $p$ -adic analytic behavior of its algebraic part viewed as a function on the weight space  $\mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$ .

**1.3. The modified Euler factors at  $p$  and  $\infty$ .** Let  $G_{\mathbf{Q}_p}$  denote the decomposition group at  $p$ . Define the rank four  $G_{\mathbf{Q}_p}$ -invariant subspace of  $\mathbf{V}$  by

$$\text{Fil}^+ \mathbf{V} := \text{Fil}^+ \mathcal{V} \otimes \omega^a \langle \varepsilon_{\text{cyc}} \rangle_T,$$

where

$$\text{Fil}^+ \mathcal{V} := \text{Fil}^0 V_f \otimes \text{Fil}^0 V_g \otimes V_h + V_f \otimes \text{Fil}^0 V_g \otimes \text{Fil}^0 V_h + \text{Fil}^0 V_f \otimes V_g \otimes \text{Fil}^0 V_h.$$

The pair  $(\text{Fil}^+ \mathbf{V}, \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}})$  satisfies the *Panchishkin condition* in [Gre94a, page 217] in the sense that for each arithmetic point  $(\underline{Q}, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$ , the Hodge-Tate numbers of  $\text{Fil}^+ \mathbf{V}_{(\underline{Q}, P)}$  are all positive, while none of the Hodge-Tate numbers of  $\mathbf{V}_{(\underline{Q}, P)}/\text{Fil}^+ \mathbf{V}_{(\underline{Q}, P)}$  is positive. Here the Hodge-Tate number of  $\mathbf{Q}_p(1)$  is one in our convention. Now we can define the modified  $p$ -Euler factor by

$$\mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{(\underline{Q}, P)}) := \frac{L_p(\text{Fil}^+ \mathbf{V}_{(\underline{Q}, P)}, 0)}{\varepsilon(\text{WD}_p(\text{Fil}^+ \mathbf{V}_{(\underline{Q}, P)})) \cdot L_p((\text{Fil}^+ \mathbf{V}_{(\underline{Q}, P)})^\vee, 1)} \cdot \frac{1}{L_p(\mathbf{V}_{(\underline{Q}, P)}, 0)}.$$

We note that this modified  $p$ -Euler factor is precisely the ratio between the factor  $\mathcal{L}_p^{(\rho)}(\mathbf{V}_{(\underline{Q}, P)})$  in [Coa89b, page 109, (18)] and the local  $L$ -factor  $L_p(\mathbf{V}_{(\underline{Q}, P)}, 0)$ .

In the theory of  $p$ -adic  $L$ -functions, we also need the modified Euler factor  $\mathcal{E}_\infty(\mathbf{V}_{(\underline{Q}, P)})$  at the archimedean place observed by Deligne. It is defined to be the ratio between the factor  $\mathcal{L}_\infty^{(\sqrt{-1})}(\mathbf{V}_{(\underline{Q}, P)})$  in [Coa89b, page 103 (4)] and the Gamma factor  $\Gamma_{\mathbf{V}_{(\underline{Q}, P)}}(0)$ . In our current case it is explicitly given by

$$\mathcal{E}_\infty(\mathbf{V}_{(\underline{Q}, P)}) = (\sqrt{-1})^{k_{Q_1} + k_{Q_2} + k_{Q_3} - 3}.$$

**1.4. Hida's canonical periods.** To give the precise definition of periods for the motive  $\mathbf{V}_{(\underline{Q}, P)}$ , we recall Hida's canonical period of an  $\mathbf{I}$ -adic primitive cuspidal Hida family  $\mathcal{F}$  of tame conductor  $N_{\mathcal{F}}$ . Let  $\mathfrak{m}_{\mathbf{I}}$  be the maximal ideal of  $\mathbf{I}$ . We say  $\mathcal{F}$  is *controllable* if the following hypothesis holds:

**Hypothesis (CR).** The residual Galois representation  $\bar{\rho}_{\mathcal{F}} := \rho_{\mathcal{F}} \pmod{\mathfrak{m}_{\mathbf{I}}} : G_{\mathbf{Q}} \rightarrow \text{GL}_2(\bar{\mathbb{F}}_p)$  is absolutely irreducible and is  $p$ -distinguished.

Suppose that  $\mathcal{F}$  is controllable. Then the local component of the universal cuspidal ordinary Hecke algebra corresponding to  $\mathcal{F}$  is known to be Gorenstein by [MW86, Prop. 2, §9] and [Wil95, Corollary 2, page 482], and with this Gorenstein property, Hida proved in [Hid88a, Theorem 0.1] that the congruence module for  $\mathcal{F}$  is isomorphic to  $\mathbf{I}/(\eta_{\mathcal{F}})$  for some non-zero element  $\eta_{\mathcal{F}} \in \mathbf{I}$  if  $p > 3$ . Moreover, for any arithmetic point  $Q \in \mathfrak{X}_{\mathbf{I}}^+$ , the specialization  $\eta_{\mathcal{F}_Q} = Q(\eta_{\mathcal{F}})$  generates the congruence ideal of  $\mathcal{F}_Q$ . We denote by  $\mathcal{F}_Q^\circ$  the normalized newform of weight  $k_Q$ , conductor  $N_Q = N_{\mathcal{F}} p^{n_Q}$  with nebentypus  $\chi_Q$  corresponding to  $\mathcal{F}_Q$ . There is a unique decomposition  $\chi_Q = \chi'_Q \chi_{Q, (p)}$ , where  $\chi'_Q$  and  $\chi_{Q, (p)}$  are Dirichlet characters modulo  $N_{\mathcal{F}}$  and  $p^{n_Q}$  respectively. Let  $\alpha_Q = \mathbf{a}(p, \mathcal{F}_Q)$ . Define the modified Euler factor  $\mathcal{E}_p(\mathcal{F}_Q, \text{Ad})$  for the adjoint motive of  $\mathcal{F}_Q$  by

$$\mathcal{E}_p(\mathcal{F}_Q, \text{Ad}) = \alpha_Q^{-2n_Q} \times \begin{cases} (1 - \alpha_Q^{-2} \chi_Q(p) p^{k_Q-1})(1 - \alpha_Q^{-2} \chi_Q(p) p^{k_Q-2}) & \text{if } n_Q = 0, \\ -1 & \text{if } n_Q = 1, \chi_{Q, (p)} = 1 \text{ (so } k_Q = 2), \\ \mathfrak{g}(\chi_{Q, (p)}) \chi_{Q, (p)}(-1) & \text{if } n_Q > 0, \chi_{Q, (p)} \neq 1. \end{cases}$$

Here  $\mathfrak{g}(\chi_{Q, (p)})$  is the usual Gauss sum. Fixing the choice of a generator  $\eta_{\mathcal{F}}$  and letting  $\|\mathcal{F}_Q^\circ\|_{\Gamma_0(N_Q)}^2$  be the usual Petersson norm of  $\mathcal{F}_Q^\circ$ , we define the *canonical period*  $\Omega_{\mathcal{F}_Q}$  of  $\mathcal{F}$  at  $Q$  by

$$\Omega_{\mathcal{F}_Q} := (-2\sqrt{-1})^{k_Q+1} \cdot \|\mathcal{F}_Q^\circ\|_{\Gamma_0(N_Q)}^2 \cdot \frac{\mathcal{E}_p(\mathcal{F}_Q, \text{Ad})}{\iota_p(\eta_{\mathcal{F}_Q})} \in \mathbf{C}^\times.$$

By [Hid16, Corollary 6.24, Theorem 6.28], one can show that for each arithmetic point  $Q$ , up to a  $p$ -adic unit, the period  $\Omega_{\mathcal{F}_Q}$  is equal to the product of the plus/minus canonical periods  $\Omega(+; \mathcal{F}_Q^\circ) \Omega(-; \mathcal{F}_Q^\circ)$  introduced in [Hid94, page 488].

1.5. **Statement of the main result.** We impose the following technical assumption:

(sf)  $N_i$  is square-free and  $\chi_i = \omega^{a_i}$  is a power of the Teichmüller character for  $i = 1, 2, 3$ .

Our main result is a construction of the balanced  $p$ -adic triple product  $L$ -functions:

**Theorem A.** *In addition to (sf), we further suppose that  $p > 3$  and that  $\mathbf{f}, \mathbf{g}$  and  $\mathbf{h}$  satisfy Hypothesis (CR). Fix generators  $(\eta_{\mathbf{f}}, \eta_{\mathbf{g}}, \eta_{\mathbf{h}})$  of the congruence ideals of  $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ . Then for each  $a \in \mathbf{Z}/(p-1)\mathbf{Z}$ , there exists a unique element  $L_{\mathbf{F},(a)}^* \in \mathbf{I}_4$  such that for each arithmetic point  $(\underline{Q}, P) = (Q_1, Q_2, Q_3, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$ , we have*

$$L_{\mathbf{F},(a)}^*(\underline{Q}, P) = \Gamma_{\mathbf{V}_{(\underline{Q}, P)}}(0) \cdot \frac{L(\mathbf{V}_{(\underline{Q}, P)}, 0)}{\Omega_{\mathbf{f}_{Q_1}} \Omega_{\mathbf{g}_{Q_2}} \Omega_{\mathbf{h}_{Q_3}}} \cdot (\sqrt{-1})^{k_{Q_1} + k_{Q_2} + k_{Q_3} - 3} \cdot \mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{(\underline{Q}, P)}).$$

In the literature, the three weight variable  $p$ -adic  $L$ -function for the triple product of Hida families in the balanced case has been extensively studied by Greenberg-Seveso [GS16], the first author [Hsi19] and so on. These works, based on Ichino's formula [Ich08], focuses on the  $p$ -adic interpolation of central values and hence the cyclotomic variable is excluded. Our four-variable  $p$ -adic  $L$ -function  $L_{\mathbf{F},(a)}^*$  specializes to this three variable  $p$ -adic  $L$ -function along the central critical line (see Remark 7.8). The first attempt to construct the cyclotomic  $p$ -adic triple product  $L$ -functions was made by Böcherer and Panchishkin [BP06, BP09], where they constructed one-variable  $p$ -adic  $L$ -functions associated with three primitive elliptic newforms. Their construction is not restricted to the ordinary case but the interpolation formula is less complete and the  $p$ -integrality of the  $p$ -adic  $L$ -function is not discussed. Without the Hypothesis (CR), we construct a canonical four-variable  $p$ -adic triple product  $L$ -functions but with denominators (see Corollary 7.9).

1.6. **Application to the trivial zero conjecture.** Let  $E_i$  be a  $p$ -ordinary elliptic curve over the rationals  $\mathbf{Q}$  of square-free conductor  $M_i$ . We write  $L(\mathbf{E}, s)$  for the degree eight motivic  $L$ -function for the triple product

$$(1.1) \quad \mathbf{V}_{\mathbf{E}} = H_{\acute{e}t}^1(E_1/\overline{\mathbf{Q}}, \mathbf{Q}_p) \otimes H_{\acute{e}t}^1(E_2/\overline{\mathbf{Q}}, \mathbf{Q}_p) \otimes H_{\acute{e}t}^1(E_3/\overline{\mathbf{Q}}, \mathbf{Q}_p)$$

realized in the middle cohomology of the abelian variety  $\mathbf{E} = E_1 \times E_2 \times E_3$  by the Künneth formula. Hence

$$L(H_{\acute{e}t}^3(\mathbf{E}/\overline{\mathbf{Q}}, \mathbf{Q}_p), s) = L(\mathbf{E}, s) \prod_{i=1}^3 L(E_i, s-1)^2.$$

Our four-variable  $p$ -adic  $L$ -function yields a cyclotomic  $p$ -adic  $L$ -function

$$L_p(\mathbf{E}) \in \mathbf{Z}_p[[\text{Gal}(\mathbf{Q}_{\infty}/\mathbf{Q})]] \otimes \mathbf{Q}_p,$$

which roughly interpolates the algebraic part of central values  $\frac{L(\mathbf{E} \otimes \chi, 2)}{\Omega}$  with a fixed period  $\Omega$  for all finite order characters  $\chi$  of  $\text{Gal}(\mathbf{Q}_{\infty}/\mathbf{Q})$ . Define an analytic function  $L_p(\mathbf{E}, s) := \varepsilon_{\text{cyc}}^{s-2}(L_p(\mathbf{E}))$  for  $s \in \mathbf{Z}_p$  (See Proposition 8.2 for the precise statement). The Euler-like factor  $\mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{\mathbf{E}}(2))$  can possibly vanish. In this case the interpolation formula forces  $L_p(\mathbf{E}, 2)$  to be zero. Such a zero is called a trivial zero. For example, it appears if all  $E_i$  have split multiplicative reduction at  $p$  (see Remark 8.3). In this particular case, the trivial zero conjecture predicts that the leading coefficient of  $L_p(\mathbf{E}, s)$  is the product of the  $\mathcal{L}$ -invariants for  $E_i$  and the algebraic part of the complex central value  $L(\mathbf{E}, 2)$  (cf. [Gre94b, (25), p. 166] and [Ben11, p. 1579]). Using the method of Greenberg-Stevens [GS93] and [BDJ17], we establish the trivial zero conjecture for the triple product of elliptic curves. The following result is a special case of our more general result (see Theorem 8.4).

**Theorem B.** *If  $E_1, E_2, E_3$  are split multiplicative at  $p$ , then  $L_p(\mathbf{E}, s)$  has at least a triple zero at  $s = 2$  and*

$$\lim_{s \rightarrow 2} \frac{L_p(\mathbf{E}, s)}{(s-2)^3} = \prod_{i=1}^3 \mathcal{L}_p(E_i) \cdot \frac{L(\mathbf{E}, 2)}{\Omega},$$

where  $\mathcal{L}_p(E_i) = \frac{\log_p q_{E_i}}{\text{ord}_p q_{E_i}}$  is the  $\mathcal{L}$ -invariant of  $E_i$  with Tate's  $p$ -adic period  $q_{E_i}$  attached to  $E_i$ .

In the case of a  $p$ -adic  $L$ -function  $L_p(E, s)$  of an elliptic curve  $E$  over  $\mathbf{Q}$  the trivial zero arises if and only if  $E$  is split multiplicative at  $p$ . An analogous formula for  $L'_p(E, 1)$  was experimentally discovered in [MTT86] and proved in [GS93], and for Hilbert modular forms in [Mok09], [Spi14] and [BDJ17]. Our result proves the first cases of the trivial zero conjecture where multiple trivial zeros are present and the Galois representation is not of  $\text{GL}(2)$ -type.

1.7. **The construction of  $L_{\mathbf{F},(a)}^*$ .** We give a sketch of the construction of  $L_{\mathbf{F},(a)}^*$ . Our method is the combination of Garrett's integral representation of the triple product  $L$ -function, an integrality result of critical  $L$ -values for triple products in [Miz90] and Hida's  $p$ -adic Rankin-Selberg method. We begin with a construction of the four-variable  $p$ -adic family of the pull-back of Siegel-Eisenstein series. For each point  $x = (Q_1, Q_2, Q_3, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$ , we reorder the weights  $\{k_{Q_1}, k_{Q_2}, k_{Q_3}\} = \{k_x, l_x, m_x\}$  so that  $k_x \geq l_x \geq m_x$ . For each  $\nu_1, \nu_2 \in \{0, 1\}$ , we put

$$\mathfrak{X}_{(\nu_1, \nu_2)}^{\text{bal}} = \left\{ x \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}} \mid k_x \equiv l_x + \nu_1 \equiv m_x + \nu_2 \pmod{2} \right\}.$$

Hence we have the partition of the weight space

$$\mathfrak{X}_{\mathbf{I}_4}^{\text{bal}} = \coprod_{\nu_1, \nu_2 \in \{0, 1\}} \mathfrak{X}_{(\nu_1, \nu_2)}^{\text{bal}}.$$

Let  $N = \text{lcm}(N_1, N_2, N_3)$ . For each  $x \in \mathfrak{X}_{(\nu_1, \nu_2)}^{\text{bal}}$ , we shall construct a nearly holomorphic Siegel-Eisenstein series  $\mathbf{E}_x^{(\nu_1, \nu_2)}(Z, s)$  of degree three, weight  $(k_x, k_x - \nu_1, k_x - \nu_2)$  and level  $\Gamma_1^{(3)}(Np^\infty)$  and consider the pull-back given by

$$G_x^{(\nu_1, \nu_2)}(z_1, z_2, z_3) := e_{\text{ord}} \text{Hol} \left( \lambda_{z_2}^{\frac{k_x - l_x - \nu_1}{2}} \lambda_{z_3}^{\frac{k_x - m_x - \nu_2}{2}} \mathbf{E}_x^{(\nu_1, \nu_2)} \left( \text{diag}(z_1, z_2, z_3), k_P - \frac{w_Q + 1}{2} \right) \right),$$

where  $\lambda_z := -\frac{1}{2\pi\sqrt{-1}}(\text{Im } z)^2 \frac{\partial}{\partial \bar{z}}$  is the weight-lowering differential operator, Hol is the holomorphic projection and  $e_{\text{ord}}$  is Hida's ordinary projector. Then we show that  $G_x^{(\nu_1, \nu_2)}$  is an ordinary cusp form of weight  $(k_x, l_x, m_x)$  on  $\mathfrak{H}_1^3$  the product of three copies of the upper half plane.

The most crucial (and perhaps surprising) point is that the four classes of Siegel-Eisenstein series  $\mathbf{E}_x^{(\nu_1, \nu_2)}$  can be constructed so that  $G_x^{(\nu_1, \nu_2)}$  can be put into a single four-variable Hida family of triple product modular forms. More precisely, let  $S^{\text{ord}}(N, \chi)$  denote the space of ordinary  $\Lambda$ -adic modular forms of tame level  $N$  and character  $\chi$ . In the following we will associate to  $a \in \mathbf{Z}/(p-1)\mathbf{Z}$  and  $\underline{\chi} = (\chi_1, \chi_2, \chi_3)$  an explicit triple product ordinary  $\Lambda$ -adic form

$$\mathcal{G}_{\underline{\chi}}^{(a)} \in S^{\text{ord}}(N, \chi_1, \mathbf{Z}_p[[X_1]]) \widehat{\otimes}_{\mathbf{Z}_p} S^{\text{ord}}(N, \chi_2, \mathbf{Z}_p[[X_2]]) \widehat{\otimes}_{\mathbf{Z}_p} S^{\text{ord}}(N, \chi_3, \mathbf{Z}_p[[X_3]]) \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p[[\text{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]].$$

By an explicit calculation of Fourier coefficients of  $G_x^{(\nu_1, \nu_2)}$ , we prove in Proposition 6.8 that the specialization  $\mathcal{G}_{\underline{\chi}}^{(a)}(x)$  at every  $x \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$  is the  $q$ -expansion of  $G_x^{(\nu_1, \nu_2)}$ .

Let  $T_3^+$  be the set of positive definite half-integral matrices of size 3. The Siegel series attached to  $B \in T_3^+$  and a rational prime  $\ell$  is defined by

$$b_\ell(B, s) = \sum_{z \in \text{Sym}_3(\mathbf{Q}_\ell) / \text{Sym}_3(\mathbf{Z}_\ell)} \psi(-\text{tr}(Bz)) \nu[z]^{-s},$$

where  $\psi$  is an arbitrarily fixed additive character on  $\mathbf{Q}_\ell$  of order 0 and  $\nu[z]$  is the product of denominators of elementary divisors of  $z$ . There exists a polynomial  $F_{B, \ell}(X) \in \mathbf{Z}[X]$  such that

$$b_\ell(T, s) = (1 - \ell^{-s})(1 - \ell^{2-2s})F_{B, \ell}(\ell^{-s}).$$

Let  $z \mapsto [z]$  denote the inclusion of group-like elements  $1 + p\mathbf{Z}_p \hookrightarrow \mathbf{Z}_p[[1 + p\mathbf{Z}_p]]^\times$ . Fix a topological generator  $\mathbf{u} \in 1 + p\mathbf{Z}_p$  and identify  $\mathbf{Z}_p[[1 + p\mathbf{Z}_p]]$  with  $\mathbf{Z}_p[[X]]$ , where  $X = [\mathbf{u}] - 1$ . Define a character  $\langle \cdot \rangle : \mathbf{Z}_p^\times \rightarrow 1 + p\mathbf{Z}_p$  by  $\langle x \rangle = x\omega(x)^{-1}$  and write  $\langle x \rangle_X = [x] = (1 + X)^{\log_p z / \log_p \mathbf{u}} \in \mathbf{Z}_p[[X]]$ . Let  $\Xi_p$  be a set of symmetric matrices of size 3 over  $\mathbf{Z}_p$  whose off-diagonal entries are  $p$ -units but whose diagonal entries are not. Now the seven-variable formal power series is presented by

$$\mathcal{G}_{\underline{\chi}}^{(a)} = \sum_{B=(b_{ij}) \in T_3^+ \cap \Xi_p} \mathcal{Q}_B^{(a)}(X_1, X_2, X_3, T) \cdot \mathcal{F}_B^{(a)}(X_1, X_2, X_3, T) \cdot q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}},$$

where  $\mathcal{Q}_B^{(a)}, \mathcal{F}_B^{(a)} \in \mathbf{Z}_p[[X_1, X_2, X_3, T]]$  are given by

$$\begin{aligned} \mathcal{Q}_B^{(a)}(X_1, X_2, X_3, T) &= \frac{\omega^a(8b_{23}b_{31}b_{12}) \langle 8b_{23}b_{31}b_{12} \rangle_T}{\chi_1(2b_{23})\chi_2(2b_{31})\chi_3(2b_{12}) \langle 2b_{23} \rangle_{X_1} \langle 2b_{31} \rangle_{X_2} \langle 2b_{12} \rangle_{X_3}}, \\ \mathcal{F}_B^{(a)}(X_1, X_2, X_3, T) &= \prod_{\ell|pN} F_{B,\ell}(\langle \ell \rangle_T^{-2} (\omega^{-2a}\chi_1\chi_2\chi_3)(\ell) \langle \ell \rangle_{X_1} \langle \ell \rangle_{X_2} \langle \ell \rangle_{X_3} \ell^{-4}). \end{aligned}$$

Now we apply the  $p$ -adic Rankin-Selberg method to define the  $p$ -adic  $L$ -function. Denote the universal ordinary cuspidal Hecke algebra by  $T(N, \chi, \mathbf{I})$ . For each  $?\in\{\mathbf{f}, \mathbf{g}, \mathbf{h}\}$  we write  $1_? \in T(N_1, \chi_1, \mathbf{I}) \otimes_{\mathbf{I}} \text{Frac } \mathbf{I}$  for the idempotent corresponding to  $?$ . We define

$$L_{\mathbf{F},(a)} := \text{the first Fourier coefficient of } \eta_{\mathbf{f}}\eta_{\mathbf{g}}\eta_{\mathbf{h}}(\mathbf{1}_{\mathbf{f}} \otimes \mathbf{1}_{\mathbf{g}} \otimes \mathbf{1}_{\mathbf{h}}(\text{Tr}_{N/N_1} \otimes \text{Tr}_{N/N_2} \otimes \text{Tr}_{N/N_3}(\mathcal{G}_{\underline{\chi}}^{(a)})) \in \mathbf{I}_3[[T]],$$

where  $\text{Tr}_{N/N_i} : S^{\text{ord}}(N, \chi_i, \mathbf{I}) \rightarrow S^{\text{ord}}(N_i, \chi_i, \mathbf{I})$  is the usual trace map, and then the  $p$ -adic triple product  $L$ -function is defined to be  $L_{\mathbf{F},(a)}^* = L_{\mathbf{F},(a)} \cdot \mathfrak{f}_{\underline{\chi}, a, N_1, N_2, N_3}^{-1}$ , where  $\mathfrak{f}_{\underline{\chi}, a, N_1, N_2, N_3} \in \mathbf{I}_4^\times$  is a fudge factor which is essentially a product of epsilon factors at prime-to- $p$  finite places. The  $p$ -adic Rankin-Selberg method tells us that the interpolation formula for the value  $L_{\mathbf{F},(a)}(x)$  at  $x \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$  is roughly given by

$$\lim_{s \rightarrow k_P - \frac{w_{\mathcal{Q}} + 1}{2}} \eta_{\mathbf{f}_{Q_1}} \eta_{\mathbf{g}_{Q_2}} \eta_{\mathbf{h}_{Q_3}} \frac{\langle \mathbf{f}_{Q_1} \otimes \mathbf{g}_{Q_2} \otimes \mathbf{h}_{Q_3}, \mathbf{E}_x^{(\nu_1, \nu_2)}(s) \rangle}{\|\mathbf{f}_{Q_1}\|^2 \|\mathbf{g}_{Q_2}\|^2 \|\mathbf{h}_{Q_3}\|^2}$$

(cf. Lemma 7.3), where  $\langle \cdot, \cdot \rangle$  is the Petersson pairing on  $\mathfrak{H}_1^3$  and  $\|\cdot\|$  is the Petersson norm on  $\mathfrak{H}_1$ . The series  $\mathbf{E}_x^{(\nu_1, \nu_2)}(Z, s)$  is constructed from a factorizable section of a certain family of induced representations. By means of the generalization of Garrett's work, carried out in [PSR87, Ike89] (see Lemma 7.1) the pairing can be unfolded and written as a product of  $L(s + \frac{1}{2}, \pi_{\mathbf{f}_{Q_1}} \times \pi_{\mathbf{g}_{Q_2}} \times \pi_{\mathbf{h}_{Q_3}} \otimes \epsilon_P \omega^{\alpha - k_P})$  and the normalized local zeta integrals at primes dividing  $pN$ . It turns out that these local zeta integrals are essentially given by the modified Euler factor  $\mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{(\underline{Q}, P)})$  at  $p$  and the local epsilon factors  $\mathfrak{f}_{\underline{\chi}, a, N_1, N_2, N_3}$  at primes  $\ell|N$ . In both calculations the key ingredients are Lemma 2.1 and the local functional equations for  $\text{GL}_1$  and  $\text{GL}_2$ , by which we can generalize Proposition 4.2 of [GK92] without brute force calculations (see Remark 3.3).

This paper is organized as follows. In §2, §3 and §4, we make the choices of local datum for Siegel Eisenstein series  $\mathbf{E}_x^{(\nu_1, \nu_2)}(Z, s)$  and carry out the explicit computation of local zeta integrals that appear in Garrett's integral representation of triple product  $L$ -functions. After preparing some notation in Hida theory in §5, we show that the Fourier expansion of  $G_x^{(\nu_1, \nu_2)}$  can be  $p$ -adically interpolated by the power series  $\mathcal{G}_{\underline{\chi}}^{(a)}$  in §6. The key ingredient is Proposition 6.3 about the computation of Fourier coefficients of  $G_x^{(\nu_1, \nu_2)}$ . In §7, we put all the local computations in §2, 3, and 4 and prove the main interpolation formulae in Theorem 7.6. Finally, in §8 we construct some improved  $p$ -adic  $L$ -functions in Lemmas 8.5 and 8.6 and use them to prove the trivial zero conjecture for the triple product of elliptic curves in Theorem 8.4.

**Notation.** The following notations will be used frequently throughout the paper. For an associative ring  $R$  with identity element, we denote by  $R^\times$  the group of all its invertible elements, and by  $\text{M}_{m,n}(R)$  the module of all  $m \times n$  matrices with entries in  $R$ . Put  $\text{M}_n(R) = \text{M}_{n,n}(R)$  and  $\text{GL}_n(R) = \text{M}_n(R)^\times$  particularly when we view the set as a ring. The identity and zero elements of the ring  $\text{M}_n(R)$  are denoted by  $\mathbf{1}_n$  and  $\mathbf{0}_n$  (when  $n$  needs to be stressed) respectively. The transpose of a matrix  $x$  is denoted by  $x^t$ . Let  $\text{Sym}_n(R) = \{z \in \text{M}_n(R) \mid z^t = z\}$  be the space of symmetric matrices of size  $n$  over  $R$ . For any set  $X$  we denote by  $\mathbb{1}_X$  the characteristic function of  $X$ . When  $X$  is a finite set, we denote by  $\sharp X$  the number of elements in  $X$ . When  $X$  is a totally disconnected locally compact topological space or a smooth real manifold, we write  $\mathcal{S}(X)$  for the space of Schwartz-Bruhat functions on  $X$ . If  $x$  is a real number, then we put  $\lceil x \rceil = \max\{i \in \mathbf{Z} \mid i \leq x\}$ .

If  $R$  is a commutative ring and  $G = \text{GL}_2(R)$ , we denote by  $\rho$  the right translation of  $G$  on the space of  $\mathbf{C}$ -valued functions on  $G$ . Thus  $(\rho(g)f)(g') = f(g'g)$ . We write  $\mathbf{1} : G \rightarrow \mathbf{C}$  for the constant function  $\mathbf{1}(g) = 1$ . For a function  $f : G \rightarrow \mathbf{C}$  and a character  $\chi : R^\times \rightarrow \mathbf{C}^\times$ , let  $f \otimes \chi : G \rightarrow \mathbf{C}$  denote the function  $f \otimes \chi(g) = f(g)\chi(\det g)$ .

## 2. COMPUTATION OF THE LOCAL ZETA INTEGRAL: THE $p$ -ADIC CASE

**2.1. The local zeta integral.** Let  $T_n$  be the subgroup of diagonal matrices in  $\text{GL}_n$ ,  $U_n$  the subgroup of upper triangular unipotent matrices in  $\text{GL}_n$ ,  $Z_n$  the subgroup of scalar matrices in  $\text{GL}_n$  and  $B_n = T_n U_n$  the

standard Borel subgroup of  $\mathrm{GL}_n$ . The symplectic similitude group of degree  $n$  is defined by

$$\mathrm{GSp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid gJ_n g^t = \nu_n(g)J_n, \nu_n(g) \in \mathrm{GL}_1\}, \quad J_n = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}.$$

We define the homomorphisms

$$\mathbf{m} : \mathrm{GL}_n \times \mathrm{GL}_1 \rightarrow \mathrm{GSp}_{2n}, \quad \mathbf{n}, \mathbf{n}^- : \mathrm{Sym}_n \rightarrow \mathrm{GSp}_{2n}$$

by

$$\mathbf{m}(A, \nu) = \begin{pmatrix} A & 0 \\ 0 & \nu(A^t)^{-1} \end{pmatrix}, \quad \mathbf{n}(z) = \begin{pmatrix} \mathbf{1}_n & z \\ 0 & \mathbf{1}_n \end{pmatrix}, \quad \mathbf{n}^-(z) = \begin{pmatrix} \mathbf{1}_n & 0 \\ z & \mathbf{1}_n \end{pmatrix}.$$

We write

$$\mathbf{m}(A) = \mathbf{m}(A, 1), \quad \mathbf{d}(\nu) = \mathbf{m}(\mathbf{1}_n, \nu).$$

A maximal parabolic subgroup  $\mathcal{P}_n = \mathcal{M}_n N_n$  of  $\mathrm{GSp}_{2n}$  is defined by

$$\mathcal{M}_n = \mathbf{m}(\mathrm{GL}_n \times \mathrm{GL}_1), \quad N_n = \mathbf{n}(\mathrm{Sym}_n).$$

Define algebraic groups of  $U^0 \subset U \subset H$  by

$$\begin{aligned} H &= \{(g_1, g_2, g_3) \in (\mathrm{GL}_2)^3 \mid \det g_1 = \det g_2 = \det g_3\}, \\ U &= \{(\mathbf{n}(x_1), \mathbf{n}(x_2), \mathbf{n}(x_3)) \mid x_1, x_2, x_3 \in \mathbf{M}_1\}, \\ U^0 &= \{(\mathbf{n}(x_1), \mathbf{n}(x_2), \mathbf{n}(x_3)) \mid x_1 + x_2 + x_3 = 0\}. \end{aligned}$$

We define the embedding  $\iota : H \hookrightarrow \mathrm{GSp}_6$  by

$$\iota \left( \left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \right) \right) = \left( \begin{array}{cc|cc} a_1 & & b_1 & \\ & a_2 & & b_2 \\ \hline & & a_3 & b_3 \\ c_1 & & & d_1 \\ & c_2 & & d_2 \\ \hline & & c_3 & d_3 \end{array} \right).$$

We identify  $Z = Z_6$  with the center of  $\mathrm{GSp}_6$ . Put

$$\eta = \left( \begin{array}{ccc|ccc} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array} \right).$$

Let  $F$  be a local field of characteristic zero. In the nonarchimedean case  $F$  contains a ring  $\mathfrak{o}$  of integers having a single prime ideal  $\mathfrak{p}$  and the absolute value  $\alpha_F = |\cdot|$  on  $F$  is normalized via  $|\varpi| = q^{-1}$  for any generator  $\varpi$  of  $\mathfrak{p}$ , where  $q$  denotes the order of the residue field  $\mathfrak{o}/\mathfrak{p}$ . Fix an additive character  $\psi$  on  $F$  which is trivial on  $\mathfrak{o}$  but non-trivial on  $\mathfrak{p}^{-1}$ . When  $F = \mathbf{R}$ , we define  $\psi(x) = e^{2\pi\sqrt{-1}x}$  for  $x \in \mathbf{R}$ .

Let  $K$  be a standard maximal compact subgroup of  $\mathrm{GSp}_6(F)$ . For quasi-characters  $\hat{\omega}, \chi : F^\times \rightarrow \mathbf{C}^\times$  we let  $I_3(\hat{\omega}, \chi) := \mathrm{Ind}_{\mathcal{P}_3}^{\mathrm{GSp}_6(F)} \chi^2 \hat{\omega} \boxtimes \chi^{-3} \hat{\omega}^{-1}$  be the space of all right  $K$ -finite functions  $f$  on  $\mathrm{GSp}_6(F)$  which satisfy

$$f(\mathbf{m}(A, \lambda)\mathbf{n}(z)g) = \hat{\omega}(\lambda^{-2} \det A) \chi(\lambda^{-3}(\det A)^2) |\lambda^{-3}(\det A)^2| f(g)$$

for  $A \in \mathrm{GL}_3(F)$ ,  $\lambda \in F^\times$ ,  $z \in \mathrm{Sym}_3(F)$  and  $g \in \mathrm{GSp}_6(F)$ . The group  $\mathrm{GSp}_6(F)$  acts on  $I_3(\hat{\omega}, \chi)$  by right translation  $\rho_3$ . It is important to note that for  $t = \mathrm{diag}(a, d) \in T_2$

$$(2.1) \quad f(\eta \nu(tg_1, tg_2, tg_3)) = \hat{\omega}(d)^{-1} \chi(ad^{-1}) |ad^{-1}| f(\eta \nu(g_1, g_2, g_3)).$$

It is well worthy of notice that

$$(2.2) \quad I_3(\hat{\omega}, \chi) \otimes \mu \circ \nu_3 = I_3(\hat{\omega} \mu^{-2}, \chi \mu).$$

We call a  $K$ -finite function  $(s, g) \mapsto f_s(g)$  on  $\mathbf{C} \times \mathrm{GSp}_6(F)$  a holomorphic section of  $I_3(\hat{\omega}, \chi \alpha_F^s)$  if  $f_s(g)$  is holomorphic in  $s$  for each  $g \in \mathrm{GSp}_6(F)$  and  $f_s \in I_3(\hat{\omega}, \chi \alpha_F^s)$  for each  $s \in \mathbf{C}$ . We associate to a non-degenerate symmetric matrix  $B$  of size 3 the degenerate Whittaker functional

$$\mathcal{W}_B : I_3(\hat{\omega}, \chi \alpha_F^s) \rightarrow \mathbf{C}, \quad \mathcal{W}_B(f_s) = \int_{\mathrm{Sym}_3(F)} f_s(J_3 \mathbf{n}(z)) \psi(-\mathrm{tr}(Bz)) dz.$$

The integral converges if  $\mathrm{Re} s$  is sufficiently large and can be continued to an entire function.

Given an irreducible admissible infinite dimensional representation  $\pi$  of  $\mathrm{GL}_2(F)$ , we denote by  $\mathscr{W}(\pi)$  the Whittaker model of  $\pi$  with respect to  $\psi$ . Let  $\pi_1, \pi_2, \pi_3$  be a triplet of irreducible admissible infinite dimensional representations of  $\mathrm{GL}_2(F)$ . We denote the central character of  $\pi_i$  by  $\omega_i$ . Set  $\hat{\omega} = \omega_1 \omega_2 \omega_3$ . We associate to a holomorphic section  $f_s$  of  $I(\hat{\omega}, \chi \alpha_F^s)$  and Whittaker functions  $W_i \in \mathscr{W}(\pi_i)$  the local zeta integral

$$Z(W_1, W_2, W_3, f_s) = \int_{U^0 Z \backslash H} W_1(g_1) W_2(g_2) W_3(g_3) f_s(\boldsymbol{\eta} \nu(g_1, g_2, g_3)) dg_1 dg_2 dg_3,$$

which converges absolutely if  $\mathrm{Re} s$  is sufficiently large.

We define a map  $\iota_0 : H \hookrightarrow \mathrm{GSp}_6$  by

$$\iota_0(g_1, g_2, g_3) = \boldsymbol{\eta} \nu(g_1, g_2 J_1, g_3 J_1).$$

As a preliminary step, we choose a coordinate system on an open dense subset of  $U^0 Z \backslash H$ .

**Lemma 2.1.** *If  $(x_1, u_1, u_2, u_3, a_2, a_3) \in F^4 \oplus F^{\times 2}$ , then*

$$\iota_0(\mathbf{n}^-(u_1) \mathbf{n}(x_1), \mathbf{m}(a_2) \mathbf{n}^-(u_2), \mathbf{m}(a_3) \mathbf{n}^-(u_3)) = \begin{pmatrix} A & B \\ \mathbf{0}_3 & (A^t)^{-1} \end{pmatrix} J_3 \mathbf{n}(-z),$$

where

$$A = \begin{pmatrix} 1 & a_2 u_1 & a_3 u_1 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad B = \begin{pmatrix} -u_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} -x_1 & a_2 & a_3 \\ a_2 & u_2 + a_2^2 u_1 & a_2 a_3 u_1 \\ a_3 & a_2 a_3 u_1 & u_3 + a_3^2 u_1 \end{pmatrix}.$$

PROOF. We can prove Lemma 2.1 by the matrix expression of  $\iota_0$ . □

**2.2. The unramified case.** When  $\pi_i$  is unramified, we write  $W_i^0 \in \mathscr{W}(\pi_i)$  for the unique Whittaker function which takes the value 1 on  $\mathrm{GL}_2(\mathfrak{o})$ . Assume that  $\hat{\omega}$  and  $\chi$  are unramified. Then we define the holomorphic section  $f_s^0(\chi)$  of  $I_3(\hat{\omega}, \chi \alpha_F^s)$  by the condition that  $f_s^0(k, \chi) = 1$  for  $k \in \mathrm{GSp}_6(\mathfrak{o})$ . Garrett has proved that

$$Z(W_1^0, W_2^0, W_3^0, f_s^0(\chi)) = \frac{L(s + \frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi)}{L(2s + 2, \chi^2 \hat{\omega}) L(4s + 2, \chi^4 \hat{\omega}^2)}.$$

We associate to a half-integral symmetric matrix  $B$  the series defined by

$$b(B, s) = \sum_{z \in \mathrm{Sym}_3(F) / \mathrm{Sym}_3(\mathfrak{o})} \psi(-\mathrm{tr}(Bz)) \nu[z]^{-s},$$

where  $\psi$  is an arbitrarily fixed additive character on  $F$  of order 0 and  $\nu[z] = [z \mathfrak{o}^3 + \mathfrak{o}^3 : \mathfrak{o}^3]$ . If  $\det B \neq 0$ , then there exists a polynomial  $F_B(X) \in \mathbf{Z}[X]$  such that

$$b(B, s) = (1 - q^{-s})(1 - q^{2-2s}) F_B(q^{-s}).$$

The following relation is well-known (cf. [Shi97, Proposition 19.2, page 158]):

$$(2.3) \quad \mathcal{W}_B(f_s^0(\chi)) = \frac{F_B(\chi^2 \hat{\omega}(\varpi) q^{-2s-2})}{L(2s + 2, \chi^2 \hat{\omega}) L(4s + 2, \chi^4 \hat{\omega}^2)}.$$



**2.3. The  $p$ -adic case.** Let  $\text{St}$  stand for the Steinberg representation of  $\text{GL}_2(F)$ . For quasi-characters  $\mu, \nu$  of  $F^\times$  the representation  $I(\mu, \nu)$  is realized on the space of functions  $f : \text{GL}_2(F) \rightarrow \mathbf{C}$  which satisfy

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \mu(a)\nu(d) \left|\frac{a}{d}\right|^{1/2} f(g)$$

for  $a, d \in F^\times$ ,  $b \in F$  and  $g \in \text{GL}_2(F)$ , where  $\text{GL}_2(F)$  acts by right translation  $\rho$ . Hereafter we assume that  $\pi_i$  are not supercuspidal and are infinite dimensional. Then  $\pi_i$  is a quotient of a principal series representation  $I(\mu_i, \nu_i)$  with quasi-characters  $\mu_i, \nu_i$ . If  $\mu_i \nu_i^{-1} \neq \alpha_F^{-1}$ , then  $\pi_i \simeq I(\mu_i, \nu_i)$ . If  $\mu_i \nu_i^{-1} = \alpha_F^{-1}$ , then  $\pi_i \simeq \text{St} \otimes \mu_i \alpha_F^{1/2}$ . Let  $W_i^{\text{ord}} \in \mathcal{W}(\pi_i)$  be the unique Whittaker function characterized by

$$W_i^{\text{ord}}(\mathbf{t}(a)) = \nu_i(a) |a|^{1/2} \mathbb{I}_{\mathfrak{o}}(a)$$

for  $a \in F^\times$ , where  $\mathbf{t}(a) = \text{diag}(a, 1)$ . Fix a prime element  $\varpi$  of  $\mathfrak{o}$ . For each non-negative integer  $n$  we put

$$m_n = \mathbf{m}(\varpi^n), \quad t_n = J_1^{-1} m_n, \quad W_i^{(n)} = \pi_i(t_n) W_i^{\text{ord}}.$$

Given a character  $\mu$  of  $\mathfrak{o}^\times$ , we define  $\varphi_\mu \in \mathcal{S}(F)$  by

$$\varphi_\mu(x) = \mu(x) \mathbb{I}_{\mathfrak{o}^\times}(x).$$

We write  $c(\mu)$  for the smallest integer  $n$  such that  $\mu$  is trivial on  $\mathfrak{o}^\times \cap (1 + \mathfrak{p}^n)$ . Define the open compact subgroup  $K_0^{(g)}(\mathfrak{p}^n)$  of  $\text{GSp}_{2g}(F)$  by

$$K_0^{(g)}(\mathfrak{p}^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GSp}_{2g}(\mathfrak{o}) \mid c \in \mathfrak{M}_g(\mathfrak{p}^n) \right\}.$$

We can define characters  $\mu^\uparrow$  and  $\mu^\downarrow$  of  $K_0^{(1)}(\mathfrak{p}^n)$  by

$$(2.4) \quad \mu^\uparrow \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mu(a), \quad \mu^\downarrow \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mu(d),$$

provided that  $n \geq c(\mu)$ . We define the Fourier transform of  $\Phi \in \mathcal{S}(\text{Sym}_g(F))$  with respect to  $\psi$  by

$$\widehat{\Phi}(w) = \int_{\text{Sym}_g(F)} \Phi(z) \psi(\text{tr}(zw)) dz.$$

Given a Schwartz function  $\Phi \in \mathcal{S}(\text{Sym}_3(F))$ , we can define a section  $f_\Phi(\chi)$  of  $I_3(\widehat{\omega}, \chi)$  by requiring that

$$(2.5) \quad f_\Phi(J_3 \mathbf{n}(z), \chi) = \Phi(z)$$

for  $z \in \text{Sym}_3(F)$ . Lemma 2.1 gives

$$(2.6) \quad f_\Phi(\iota_0(\mathbf{n}^-(u_1) \mathbf{n}(x_1), \mathbf{m}(a_2) \mathbf{n}^-(u_2), \mathbf{m}(a_3) \mathbf{n}^-(u_3)), \chi) \\ = \widehat{\omega}(a_2 a_3) \chi(a_2 a_3)^2 |a_2 a_3|^2 \Phi \left( \begin{pmatrix} x_1 & -a_2 & -a_3 \\ -a_2 & -u_2 - a_2^2 u_1 & -a_2 a_3 u_1 \\ -a_3 & -a_2 a_3 u_1 & -u_3 - a_3^2 u_1 \end{pmatrix} \right).$$

Now we define  $\Phi \in \mathcal{S}(\text{Sym}_3(F))$  by

$$(2.7) \quad \Phi \left( \begin{pmatrix} u_1 & x_3 & x_2 \\ x_3 & u_2 & x_1 \\ x_2 & x_1 & u_3 \end{pmatrix} \right) = \prod_{i=1}^3 \phi_i(u_i) \varphi_i(x_i),$$

where we define  $\phi_1, \phi_2, \phi_3, \varphi_1, \varphi_2, \varphi_3 \in \mathcal{S}(F)$  by

$$\phi_1 = \phi_2 = \phi_3 = \widehat{\mathbb{I}_{\mathfrak{p}}}, \quad \varphi_1 = \widehat{\varphi_{\chi \mu_1 \nu_2 \nu_3}}, \quad \varphi_2 = \widehat{\varphi_{\chi \nu_1 \mu_2 \nu_3}}, \quad \varphi_3 = \widehat{\varphi_{\chi \nu_1 \nu_2 \mu_3}}.$$

**Lemma 2.2.** *If  $n \geq \max\{1, c(\chi), c(\mu_i), c(\nu_i) \mid i = 1, 2, 3\}$ , then*

$$\rho_3(\iota(g_1, g_2, g_3)) f_\Phi(\chi) = f_\Phi(\chi) \prod_{i=1}^3 \mu_i^\uparrow(g_i)^{-1} \nu_i^\downarrow(g_i)^{-1}$$

for  $g_1, g_2, g_3 \in K_0^{(1)}(\mathfrak{p}^{2n})$  with  $\det g_1 = \det g_2 = \det g_3$ .

PROOF. One can easily check that

$$(2.8) \quad \widehat{\varphi}_\mu \in \mathcal{S}(\mathfrak{p}^{-c(\mu)}), \quad \widehat{\varphi}_\mu(ax) = \mu(a)^{-1} \widehat{\varphi}_\mu(x), \quad \widehat{\varphi}_\mu(x+b) = \widehat{\varphi}_\mu(x)$$

for  $a \in \mathfrak{o}^\times$ ,  $b \in \mathfrak{o}$  and  $x \in F$ . Simply because  $\phi_i = \mathbb{I}_{\mathfrak{p}^{-1}}$ , we see that  $\Phi(z+c) = \Phi(z)$  for  $c \in \text{Sym}_3(\mathfrak{o})$ , which means that  $f_\Phi(\chi)$  is fixed by the action of  $\mathfrak{n}(\text{Sym}_3(\mathfrak{o}))$ . Put

$$\chi_1 = \chi \mu_1 \nu_2 \nu_3, \quad \chi_2 = \chi \nu_1 \mu_2 \nu_3, \quad \chi_3 = \chi \nu_1 \nu_2 \mu_3.$$

Since

$$\begin{pmatrix} \frac{1}{a_1} & & \\ & \frac{1}{a_2} & \\ & & \frac{1}{a_3} \end{pmatrix} \begin{pmatrix} u_1 & x_2 & x_3 \\ x_2 & u_2 & x_1 \\ x_3 & x_1 & u_3 \end{pmatrix} \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix} = \begin{pmatrix} \frac{d_1 u_1}{a_1} & \frac{d_2 x_2}{a_1} & \frac{d_3 x_3}{a_1} \\ \frac{d_1 x_2}{a_2} & \frac{d_2 u_2}{a_2} & \frac{d_3 x_1}{a_2} \\ \frac{d_1 x_3}{a_3} & \frac{d_2 x_1}{a_3} & \frac{d_3 u_3}{a_3} \end{pmatrix},$$

if  $a_i, d_i \in \mathfrak{o}^\times$  and  $\lambda = a_1 d_1 = a_2 d_2 = a_3 d_3$ , then by (2.8)

$$\begin{aligned} & \rho_3(\iota(\text{diag}(a_1, d_1), \text{diag}(a_2, d_2), \text{diag}(a_3, d_3))) f_\Phi(\chi) \\ &= \widehat{\omega} \left( \frac{d_1 d_2 d_3}{\lambda^2} \right) \chi \left( \frac{(d_1 d_2 d_3)^2}{\lambda^3} \right) \chi_1 \left( \frac{a_2}{d_3} \right) \chi_2 \left( \frac{a_1}{d_3} \right) \chi_3 \left( \frac{a_1}{d_2} \right) f_\Phi(\chi) \\ &= \widehat{\omega} \left( \frac{d_1 d_2 d_3}{\lambda^2} \right) (\mu_1 \nu_2 \nu_3) \left( \frac{a_2}{d_3} \right) (\nu_1 \mu_2 \nu_3) \left( \frac{a_1}{d_3} \right) (\nu_1 \nu_2 \mu_3) \left( \frac{a_1}{d_2} \right) f_\Phi(\chi) \\ &= \widehat{\omega} \left( \frac{d_1 d_2 d_3}{\lambda^2} \right) \omega_1 \left( \frac{a_2}{d_3} \right) \omega_2 \left( \frac{a_1}{d_3} \right) \omega_3 \left( \frac{a_1}{d_2} \right) f_\Phi(\chi) \prod_{i=1}^3 \nu_i \left( \frac{a_i}{d_i} \right) \\ &= f_\Phi(\chi) \prod_{i=1}^3 \omega_i(a_i)^{-1} \nu_i \left( \frac{a_i}{d_i} \right) = f_\Phi(\chi) \prod_{i=1}^3 \mu_i(a_i)^{-1} \nu_i(d_i)^{-1}. \end{aligned}$$

Let  $w \in \text{Sym}_3(\mathfrak{p}^{2n})$ . If  $f_\Phi(g\mathfrak{n}^-(w), \chi) \neq 0$ , then since  $g\mathfrak{n}^-(w) \in \mathcal{P}_3 J_3 \mathfrak{n}(z)$  with  $z \in \text{Sym}_3(\mathfrak{p}^{-n})$  and since

$$\mathfrak{n}(z) \mathfrak{n}^-(-w) = \begin{pmatrix} \mathbf{1}_3 - zw & \mathbf{0}_3 \\ -w & (\mathbf{1}_3 - wz)^{-1} \end{pmatrix} \mathfrak{n}((\mathbf{1}_3 - zw)^{-1} z),$$

we have  $g \in \mathcal{P}_3 J_3 \mathfrak{n}(\text{Sym}_3(\mathfrak{p}^{-n}))$ . We see by the identity above that

$$f_\Phi(J_3 \mathfrak{n}(z) \mathfrak{n}^-(w), \chi) = f_\Phi(J_3 \mathfrak{n}((\mathbf{1}_3 + zw)^{-1} z), \chi) = f_\Phi(J_3 \mathfrak{n}(z), \chi)$$

for  $z \in \text{Sym}_3(\mathfrak{p}^{-n})$  and  $w \in \text{Sym}_3(\mathfrak{p}^{2n})$ . We conclude that  $f_\Phi(\chi)$  is fixed by right translation by  $\mathfrak{n}^-(\text{Sym}_3(\mathfrak{p}^{2n}))$ . The proof is complete by  $K_0^{(1)}(\mathfrak{p}^m) = \mathfrak{n}(\mathfrak{o}) \mathfrak{d}(\mathfrak{o}^\times) \mathfrak{m}(\mathfrak{o}^\times) \mathfrak{n}^-(\mathfrak{p}^m)$ .  $\square$

#### 2.4. The $\mathfrak{p}$ -adic zeta integral.

**Proposition 2.3.** *If  $n \geq \max\{1, c(\chi), c(\mu_i), c(\nu_i) \mid i = 1, 2, 3\}$ , then*

$$\begin{aligned} Z(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, f_\Phi(\chi)) &= (1 + q^{-1})^{-3} \prod_{j=1}^3 \left( \frac{\beta_j}{q \alpha_j} \right)^n \\ &\quad \times (\chi \nu_1 \nu_2 \nu_3)(-1) \gamma \left( \frac{1}{2}, \pi_1 \otimes \chi \nu_2 \nu_3, \psi \right)^{-1} \prod_{i=2,3} \gamma \left( \frac{1}{2}, \chi \nu_1 \nu_i \mu_{5-i}, \psi \right)^{-1}, \end{aligned}$$

where  $\alpha_i = \mu_i(\varpi)$  and  $\beta_i = \nu_i(\varpi)$ .

PROOF. We associate to  $f_i \in I(\mu_i, \nu_i)$  a function  $W(f_i) \in \mathscr{W}(\pi_i)$  by

$$W(g, f_i) = \int_F f_i(J_1 \mathfrak{n}(u)g) \psi(-u) du = \lim_{k \rightarrow \infty} \int_{\mathfrak{p}^{-k}} f_i(J_1 \mathfrak{n}(u)g) \psi(-u) du.$$

Here the limit stabilizes and the integral makes sense for any  $f_i \in \pi_i$ . The integral  $W$  factors through the quotient  $I(\mu_i, \nu_i) \twoheadrightarrow \text{St} \otimes \mu_i \alpha_F^{1/2}$  when  $\mu_i \nu_i^{-1} = \alpha_F^{-1}$ . Let  $f_i^{\text{ord}} \in I(\mu_i, \nu_i)$  be such that  $f_i^{\text{ord}}(g) = 0$  unless  $g \in B_2 J_1 U_2$  and such that  $f_i^{\text{ord}}(J_1 \mathfrak{n}(x)) = \mathbb{I}_{\mathfrak{o}}(x)$  for  $x \in F$ . One can easily check

$$W_i^{\text{ord}} = W(f_i^{\text{ord}}).$$

For  $i = 2, 3$  we put

$$f'_i = \rho(m_n) f_i^{\text{ord}} \in I(\mu_i, \nu_i), \quad W'_i = W(f'_i).$$

Recall that  $m_n = \text{diag}(\varpi^n, \varpi^{-n})$ . Then

$$Z(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, f) = \int_{U^0 Z \backslash H} W_1^{(n)}(g_1) W_2'(g_2) W_3'(g_3) f(\iota_0(g_1, g_2, g_3)) dg_1 dg_2 dg_3.$$

Observe that

$$W_1^{(n)}(g_1) W_2'(g_2) W_3'(g_3) = \int_{U^0} \mathbf{W}_1(u_0(g_1, g_2, g_3); f'_2, f'_3) du_0,$$

where

$$\mathbf{W}_1(g_1, g_2, g_3; f'_2, f'_3) = W_1^{(n)}(g_1) f'_2(J_1 g_2) f'_3(J_1 g_3).$$

Substituting this expression, we are led to

$$Z(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, f_\Phi(\chi)) = \int_{Z \backslash H} \mathbf{W}_1(g; f'_2, f'_3) f_\Phi(\iota_0(g), \chi) dg.$$

Define a function  $\mathcal{F}$  on  $\text{SL}_2(F)$  by

$$\mathcal{F}(g) = \int_{\text{SL}_2(F)^2} f'_2(J_1 g_2) f'_3(J_1 g_3) f_\Phi(\iota_0(g, g_2, g_3), \chi) dg_2 dg_3.$$

Let  $T' = \mathbf{m}(F^\times)$  be the diagonal torus of  $\text{SL}_2(F)$ . Then

$$\begin{aligned} Z(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, f_\Phi(\chi)) &= \int_{F^\times} d^\times a \int_{T' \backslash \text{SL}_2(F)} W_1^{(n)}(\mathbf{t}(a)g) \int_{\text{SL}_2(F)^2} \\ &\quad f'_2(J_1 \mathbf{t}(a)g_2) f'_3(J_1 \mathbf{t}(a)g_3) f_\Phi(\iota_0(\mathbf{t}(a)g, \mathbf{t}(a)g_2, \mathbf{t}(a)g_3), \chi) dg_2 dg_3 dg \\ (2.9) \quad &= \int_{F^\times} \int_{T' \backslash \text{SL}_2(F)} W_1^{(n)}(\mathbf{t}(a)g) \chi(a) \nu_2(a) \nu_3(a) \mathcal{F}(g) dg d^\times a \end{aligned}$$

by (2.1). To justify the manipulations we show that the integral

$$\int_{F^\times} \int_{\text{SL}_2(\mathfrak{o})} \int_F |W_1^{(n)}(\mathbf{t}(a)k) \chi(a) \nu_2(a) \nu_3(a) \mathcal{F}(\mathbf{n}(x)k)| dx dk d^\times a$$

is convergent for  $\text{Re } \chi \gg 0$ . The integral

$$\int_{F^\times} |W_1^{(n)}(\mathbf{t}(a)k) \chi(a) \nu_2(a) \nu_3(a)| d^\times a$$

is absolutely convergent. We frequently use the integration formula

$$\int_{\text{SL}_2(F)} h(g) dg = \frac{\zeta(2)}{\zeta(1)} \int_F \int_F \int_{F^\times} h(\mathbf{m}(a) \mathbf{n}^-(u) \mathbf{n}(x)) d^\times a du dx$$

for an integrable function  $h$  on  $\text{SL}_2(F)$ . Observe that

$$\begin{aligned} \mathcal{F}(g) &= \frac{\zeta(2)^2}{\zeta(1)^2} \int_{F^2} dx_2 dx_3 f'_2(J_1 \mathbf{n}(x_2)) f'_3(J_1 \mathbf{n}(x_3)) \int_{F^{\times 2}} \prod_{i=2,3} (\nu_i \mu_i^{-1})(a_i) \frac{d^\times a_i}{|a_i|} \\ &\quad \times \int_{F^2} f_\Phi(\iota_0(g, \mathbf{m}(a_2) \mathbf{n}^-(u_2) \mathbf{n}(x_2), \mathbf{m}(a_3) \mathbf{n}^-(u_3) \mathbf{n}(x_3)), \chi) du_2 du_3. \end{aligned}$$

Recall that

$$\mathbf{t}(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{m}(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad \mathbf{n}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \mathbf{n}^-(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.$$

Observe that

$$f'_i(J_1 \mathbf{n}(x)) = f_i^{\text{ord}}(J_1 m_n \mathbf{n}(x \varpi^{-2n})) = \beta_i^n \alpha_i^{-n} q^n \mathbb{I}_{\mathfrak{p}^{2n}}(x).$$

Lemma 2.2 shows that

$$\rho_3(\iota(\mathbf{1}_2, \mathbf{n}(x_2), \mathbf{n}(x_3)) J_3) f_\Phi(\chi) = \rho_3(J_3) f_\Phi(\chi) \quad (x_2, x_3 \in \mathfrak{p}^{2n}).$$

It follows that  $\mathcal{F}(g)$  equals the product of  $\frac{f'_2(J_1)f'_3(J_1)}{q^{4n}(1+q^{-1})^2}$  and

$$\int_{F^{\times 2} \oplus F^2} f_{\Phi}(t_0(g, \mathbf{m}(a_2)\mathbf{n}^-(u_2), \mathbf{m}(a_3)\mathbf{n}^-(u_3)), \chi) \prod_{i=2,3} \frac{\nu_i(a_i) d^{\times} a_i}{\mu_i(a_i) |a_i|} du_i.$$

In particular,  $\mathcal{F}(\mathbf{n}^-(u)\mathbf{n}(x))$  equals the product of  $\frac{f'_2(J_1)f'_3(J_1)}{q^{4n}(1+q^{-1})^2}$  and

$$\begin{aligned} & \int_{F^{\times 2} \oplus F^2} \Phi \left( \begin{pmatrix} x & -a_2 & -a_3 \\ -a_2 & -u_2 & -a_2 a_3 u \\ -a_3 & -a_2 a_3 u & -u_3 \end{pmatrix} \right) \prod_{i=2,3} (\hat{\omega} \chi^2 \nu_i \mu_i^{-1})(a_i) |a_i| d^{\times} a_i du_i \\ &= \int_{F^{\times 2} \oplus F^2} \varphi_1(-a_2 a_3 u) \varphi_2(-a_2) \varphi_3(-a_3) \phi_1(x) \phi_3(-u_2 - a_2^2 u) \phi_2(-u_3 - a_3^2 u) \prod_{i=2,3} (\hat{\omega} \chi^2 \nu_i \mu_i^{-1})(a_i) |a_i| d^{\times} a_i du_i \end{aligned}$$

by (2.6). Its integral over  $x, u \in F$  converges absolutely if  $\operatorname{Re} \chi$  is large.

Recall the functional equations

$$(2.10) \quad \begin{aligned} \gamma \left( \frac{1}{2}, \pi_1 \otimes \chi, \boldsymbol{\psi} \right) \int_{F^{\times}} W_1(\mathbf{t}(a)g) \chi(a) d^{\times} a &= \int_{F^{\times}} W_1(\mathbf{t}(a)J_1^{-1}g) (\chi \omega_1)^{-1}(a) d^{\times} a, \\ \gamma(s, \chi, \boldsymbol{\psi}) \int_{F^{\times}} \varphi(a) \chi(a) |a|^s d^{\times} a &= \int_{F^{\times}} \widehat{\varphi}(a) \chi(a)^{-1} |a|^{1-s} d^{\times} a \end{aligned}$$

for every  $W_1 \in \mathcal{W}(\pi_1)$  and  $\varphi \in \mathcal{S}(F)$ . It follows from (2.9) that

$$(2.11) \quad \begin{aligned} & \gamma \left( \frac{1}{2}, \pi_1 \otimes \chi \nu_2 \nu_3, \boldsymbol{\psi} \right) Z(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, f_{\Phi}(\chi)) \\ &= \int_{F^{\times}} \int_{T' \setminus \operatorname{SL}_2(F)} W_1^{(n)}(\mathbf{t}(a)J_1^{-1}g) (\chi \nu_2 \nu_3 \omega_1)(a)^{-1} \mathcal{F}(g) dg d^{\times} a \\ &= \int_{F^{\times}} \int_F W_1^{(n)}(\mathbf{t}(a)J_1^{-1}\mathbf{n}(x)) (\chi \nu_2 \nu_3 \omega_1)(a)^{-1} \mathcal{F}_{\boldsymbol{\psi}}(a, x) dx d^{\times} a, \end{aligned}$$

where

$$\mathcal{F}_{\boldsymbol{\psi}}(a, x) = (1 + q^{-1})^{-1} \int_F \mathcal{F}(\mathbf{n}^-(u)\mathbf{n}(x)) \boldsymbol{\psi}(-au) du.$$

We have seen that

$$\begin{aligned} \frac{q^{4n}(1+q^{-1})^3}{f'_2(J_1)f'_3(J_1)} \mathcal{F}_{\boldsymbol{\psi}}(a, x) &= \int_{F^{\times 2}} \hat{\omega}(a_2 a_3) \chi(a_2 a_3)^2 |a_2 a_3|^2 \varphi_3(-a_2) \varphi_2(-a_3) \\ &\quad \times \int_F \varphi_1(-a_2 a_3 u) \overline{\boldsymbol{\psi}(au)} du \phi_1(x) \prod_{i=2,3} \frac{\nu_i(a_i) d^{\times} a_i}{\mu_i(a_i) |a_i|} \int_F \phi_i(u_i) du_i \\ &= \phi_1(x) \int_{F^{\times 2}} \widehat{\varphi}_1 \left( \frac{a}{a_2 a_3} \right) \prod_{i=2,3} \widehat{\phi}_i(0) (\hat{\omega} \chi^2 \nu_i \mu_i^{-1})(a_i) \varphi_{5-i}(-a_i) d^{\times} a_i. \end{aligned}$$

If  $xu \neq -1$ , then

$$J_1 \mathbf{n}(x) \mathbf{n}^-(u) = \mathbf{m}((1+ux)^{-1}) \mathbf{n}(-(1+xu)u) J_1 \mathbf{n}((1+xu)^{-1}x),$$

which implies that

$$\rho(\mathbf{n}^-(u)) f_1^{\operatorname{ord}} = f_1^{\operatorname{ord}}, \quad \pi_1(\mathbf{n}^-(u)) W_1^{\operatorname{ord}} = W_1^{\operatorname{ord}}$$

for  $u \in \mathfrak{p}^n$ . If  $\phi_1(x) \neq 0$ , then since  $x \in \mathfrak{p}^{-1}$ ,

$$W_1^{(n)}(\mathbf{t}(a)J_1 \mathbf{n}(x)) = W_1^{\operatorname{ord}}(\mathbf{t}(a)m_n \mathbf{n}^-(\varpi^{2n}x)) = q^{-n} \beta_1^n \alpha_1^{-n} \nu_1(a) |a|^{1/2} \mathbb{I}_0(a \varpi^{2n}).$$

We conclude by (2.11) that

$$\begin{aligned}
& \gamma \left( \frac{1}{2}, \pi_1 \otimes \chi \nu_2 \nu_3, \psi \right) Z(W_1^{(n)}, W_2^{(n)}, W_3^{(n)}, f_\Phi(\chi)) \\
&= \int_{F^\times} \int_F W_1^{(n)}(\mathbf{t}(a) J_1^{-1} \mathbf{n}(x)) (\chi \nu_2 \nu_3 \omega_1)(a)^{-1} \frac{f_2'(J_1) f_3'(J_1)}{q^{4n}(1+q^{-1})^3} \mathcal{F}_\psi(a, x) \, dx d^\times a \\
&= \int_{F^\times} W_1^{(n)}(\mathbf{t}(a) J_1^{-1}) (\chi \nu_2 \nu_3 \omega_1)(a)^{-1} \frac{f_2'(J_1) f_3'(J_1)}{q^{4n}(1+q^{-1})^3} \int_F \mathcal{F}_\psi(a, x) \, dx d^\times a \\
&= \frac{f_2'(J_1) f_3'(J_1)}{q^{4n}(1+q^{-1})^3} \widehat{\phi}_1(0) \widehat{\phi}_2(0) \widehat{\phi}_3(0) \int_{F^\times} \frac{W_1^{(n)}(\mathbf{t}(a) J_1^{-1})}{(\chi \nu_2 \nu_3 \omega_1)(a)} \widehat{\varphi}_1 \left( \frac{a}{a_2 a_3} \right) d^\times a \prod_{i=2,3} (\widehat{\omega} \chi^2 \nu_i \mu_i^{-1})(a_i) \varphi_{5-i}(-a_i) d^\times a_i.
\end{aligned}$$

The last integral is equal to

$$\begin{aligned}
& \widehat{\omega}(-1) \int_{F^\times} \frac{W_1^{(n)}(\mathbf{t}(aa_2a_3) J_1)}{(\chi \nu_2 \nu_3 \omega_1)(a)} \widehat{\varphi}_1(a) d^\times a \prod_{i=2,3} (\chi \nu_i \mu_{5-i})(a_i) \varphi_{5-i}(a_i) d^\times a_i \\
&= \widehat{\omega}(-1) W_1^{(n)}(J_1) \int_{F^\times} \frac{\nu_1(aa_2a_3)}{(\chi \nu_2 \nu_3 \omega_1)(a)} |aa_2a_3|^{1/2} \mathbb{I}_0(aa_1a_2\varpi^{2n}) \varphi_{\chi \mu_1 \nu_2 \nu_3}(-a) d^\times a \prod_{i=2,3} (\chi \nu_i \mu_{5-i})(a_i) \varphi_{5-i}(a_i) d^\times a_i \\
&= \widehat{\omega}(-1) W_1^{(n)}(J_1) (\chi \mu_1 \nu_2 \nu_3)(-1) \prod_{i=2,3} \int_{F^\times} \varphi_{5-i}(a_i) (\chi \nu_i \mu_{5-i})(a_i) |a_i|^{1/2} d^\times a_i.
\end{aligned}$$

In the last line we employ the fact that if  $\varphi_{5-i}(a_i) \neq 0$ , then  $a_i \in \mathfrak{p}^{-n}$ . The proof is now complete by  $f_i'(J_1) = \beta_i^n \alpha_i^{-n} q^n$ ,  $W_1^{(n)}(J_1) = \beta_1^n \alpha_1^{-n} q^{-n}$  and the functional equation (2.10).  $\square$

**2.5. Degenerate Whittaker functions at  $\mathfrak{p}$ .** Let  $\Xi_{\mathfrak{p}}$  be a subset of  $\text{Sym}_3(F)$  which consists of symmetric matrices whose the diagonal entries belong to  $\mathfrak{p}$  and whose off-diagonal entries belong to  $\frac{1}{2}\mathfrak{o}^\times$ .

**Proposition 2.4.** *Let  $B = (b_{ij}) \in \text{Sym}_3(F)$ . Put  $y_i = b_{jk}$  whenever  $\{i, j, k\} = \{1, 2, 3\}$ . Then*

$$\mathcal{W}_B(f_\Phi(\chi)) = \chi(8y_1y_2y_3) \prod_{i=1}^3 \mu_i(2y_i) \mathbb{I}_{\mathfrak{o}^\times}(2y_i) \mathbb{I}_{\mathfrak{p}}(b_{ii}) \prod_{j \in \{1,2,3\} \setminus \{i\}} \nu_j(2y_i).$$

In particular,  $\mathcal{W}_B(f_\Phi(\chi)) \neq 0$  if and only if  $B \in \Xi_{\mathfrak{p}}$ .

**PROOF.** Observe that

$$(2.12) \quad \mathcal{W}_B(f_\Phi(\chi)) = \int_{\text{Sym}_3(F)} f_\Phi(J_3 \mathbf{n}(z), \chi) \psi(-\text{tr}(Bz)) \, dz = \widehat{\Phi}(-B)$$

for any  $\Phi \in \mathcal{S}(\text{Sym}_3(F))$ . We have

$$\widehat{\Phi} \left( - \begin{pmatrix} b_{11} & y_3 & y_2 \\ y_3 & b_{22} & y_1 \\ y_2 & y_1 & b_{33} \end{pmatrix} \right) = \prod_{i=1}^3 \widehat{\varphi}_i(-2y_i) \widehat{\phi}_i(-b_{ii}) = \varphi_{\chi \mu_1 \nu_2 \nu_3}(2y_1) \varphi_{\chi \nu_1 \mu_2 \nu_3}(2y_2) \varphi_{\chi \nu_1 \nu_2 \mu_3}(2y_3) \prod_{i=1}^3 \mathbb{I}_{\mathfrak{p}}(b_{ii})$$

by definition.  $\square$

**2.6. Restatements.** We rewrite Propositions 2.3 and 2.4 in a form which is suitable for our later discussion. Suppose that  $\pi_i$  is a subrepresentation of  $I(\mu_i, \nu_i)$  with  $\mu_i$  unramified. Thus  $\omega_i = \mu_i \nu_i$  coincides with  $\nu_i$  on  $\mathfrak{o}^\times$ . Let

$$\check{W}_i(\text{diag}(a, 1)) = \nu_i(a)^{-1} |a|^{1/2} \mathbb{I}_{\mathfrak{o}}(a).$$

**Definition 2.5.** We associate to the quadruplet of characters of  $\mathfrak{o}^\times$

$$\mathcal{D} = (\chi, \omega_1, \omega_2, \omega_3)$$

a holomorphic section  $f_{\mathcal{D},s} = f_{\Phi_{\mathcal{D}}}(\chi \widehat{\omega} \alpha_F^s)$  of  $I_3(\widehat{\omega}^{-1}, \chi \widehat{\omega} \alpha_F^s)$  by

$$\Phi_{\mathcal{D}} \left( \begin{pmatrix} u_1 & x_3 & x_2 \\ x_3 & u_2 & x_1 \\ x_2 & x_1 & u_3 \end{pmatrix} \right) = \prod_{i=1}^3 \widehat{\mathbb{I}}_{\mathfrak{p}}(u_i) \widehat{\varphi}_{\chi \omega_i}(x_i).$$

For each quadruplet  $(\chi_0, \chi_1, \chi_2, \chi_3)$  of characters of  $\mathfrak{o}^\times$ , valued in a commutative ring  $R$  we set

$$\mathcal{Q}_B(\chi_0, \chi_1, \chi_2, \chi_3) := \chi_0(8b_{12}b_{23}b_{13}) \cdot \chi_1(2b_{23})\chi_2(2b_{13})\chi_3(2b_{12})\mathbb{I}_{\mathbb{F}_p}(B).$$

Given a section  $f_s$  of  $I_3(\hat{\omega}^{-1}, \chi\hat{\omega}\alpha_F^s)$ , we are interested in the quantity

$$(2.13) \quad Z_{\mathfrak{p}}^*(f_s) = \frac{Z(\rho(t_n)\check{W}_1, \rho(t_n)\check{W}_2, \rho(t_n)\check{W}_3, f_s)}{L(s + \frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi)} \prod_{i=1}^3 \frac{\zeta(1)}{\zeta(2)} \left( \frac{\omega_i(\varpi)q}{\mu_i(\varpi)^2} \right)^n.$$

**Proposition 2.6.** *Notations and assumptions being as above, we have*

$$\rho_3(\nu(g_1, g_2, g_3))f_{\mathcal{D},s} = f_{\mathcal{D},s} \prod_{i=1}^3 \omega_i^\dagger(g_i), \quad g_1, g_2, g_3 \in K_0^{(1)}(\mathfrak{p}^{2n})$$

if  $\det g_1 = \det g_2 = \det g_3$  and  $n \geq \max\{1, c(\chi), c(\omega_i)\}$ . Moreover,

$$\mathcal{W}_B(f_{\mathcal{D},s}) = \mathcal{Q}_B(\mathcal{D}), \quad Z_{\mathfrak{p}}^*(f_{\mathcal{D},s}) = \chi(-1)E_{\mathfrak{p}}\left(s + \frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi\right),$$

where

$$E_{\mathfrak{p}}(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi)^{-1} = L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi) \gamma(s, \pi_1 \otimes \chi\mu_2\mu_3, \psi) \prod_{i=2,3} \gamma(s, \chi\mu_1\mu_i\nu_{5-i}, \psi).$$

PROOF. Since  $\omega_i$  coincides with  $\nu_i$  on  $\mathfrak{o}^\times$ , we apply Proposition 2.4 and get the formula for  $\mathcal{W}_B(f_{\mathcal{D},s})$  by replacing  $\pi_i, \mu_i, \omega_i, \nu_i, \chi$  by  $\pi_i^\vee \simeq \pi_i \otimes \omega_i^{-1}, \omega_i^{-1}, \mu_i^{-1}, \nu_i^{-1}, \chi\hat{\omega}$ , respectively. Proposition 2.3 applied to  $\check{W}_i^{\text{ord}}$  and  $I_3(\hat{\omega}^{-1}, \chi\hat{\omega})$  gives

$$\begin{aligned} Z(\rho(t_n)\check{W}_1, \rho(t_n)\check{W}_2, \rho(t_n)\check{W}_3, f_{\mathcal{D},s}) &= (1+q^{-1})^{-3} \prod_{i=1}^3 \left( \frac{\nu_i(\varpi)^{-1}}{q\mu_i(\varpi)^{-1}} \right)^n \\ &\quad \times \chi(-1) \gamma\left(s + \frac{1}{2}, \pi_1^\vee \otimes (\chi\hat{\omega})\nu_2^{-1}\nu_3^{-1}, \psi\right)^{-1} \prod_{i=2,3} \gamma\left(s + \frac{1}{2}, (\chi\hat{\omega})\nu_1^{-1}\nu_i^{-1}\mu_{5-i}^{-1}, \psi\right)^{-1}, \end{aligned}$$

from which the formula for  $Z_{\mathfrak{p}}^*(s)$  readily follows.  $\square$

We will use the following lemma to achieve the functional equation of the  $p$ -adic  $L$ -function in §7.7.

**Lemma 2.7.** *Put  $\check{\chi} = \chi^{-1}\hat{\omega}^{-1}$ . Then*

$$E_{\mathfrak{p}}(1-s, \pi_1 \times \pi_2 \times \pi_3 \otimes \check{\chi}) = \hat{\omega}(-1)E_{\mathfrak{p}}(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi)\varepsilon(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi, \psi).$$

PROOF. Since  $\pi_i \otimes \omega_i^{-1} \simeq \pi_i^\vee$ , we get

$$E_{\mathfrak{p}}(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \check{\chi})^{-1} = L(s, \pi_1^\vee \times \pi_2^\vee \times \pi_3^\vee \otimes \bar{\chi}) \gamma(s, \pi_1^\vee \otimes \overline{\chi\nu_2\nu_3}, \psi) \prod_{i=2,3} \gamma(s, \overline{\chi\nu_1\nu_i\mu_{5-i}}, \psi),$$

where  $\pi_i \simeq I(\mu_i, \nu_i)$ . By definition we arrive at

$$\begin{aligned} &\varepsilon(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi, \psi) L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi)^{-1} E_{\mathfrak{p}}(1-s, \pi_1 \times \pi_2 \times \pi_3 \otimes \check{\chi})^{-1} \\ &= \gamma(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi, \psi) \gamma(1-s, \pi_1^\vee \otimes \overline{\chi\mu_2\mu_3}, \psi) \prod_{i=2,3} \gamma(1-s, \overline{\chi\mu_1\mu_i\nu_{5-i}}, \psi). \end{aligned}$$

The statement can now be deduced from multiplicativity and the functional equation of gamma factors.  $\square$

### 3. COMPUTATION OF THE LOCAL ZETA INTEGRAL: THE RAMIFIED CASE

Recall that  $\text{St}$  denotes the Steinberg representation of  $\text{GL}_2(F)$ . Let  $\pi_i$  be either an irreducible unramified principal series representation or the Steinberg representation. Since

$$(3.1) \quad Z(W_1 \otimes \chi_1, W_2 \otimes \chi_2, W_3 \otimes \chi_3, f) = Z(W_1, W_2, W_3, f \otimes (\chi_1\chi_2\chi_3) \circ \nu_3)$$

for characters  $\chi_1, \chi_2, \chi_3$  of  $F^\times$ , where

$$(W_i \otimes \chi_i)(g_i) = W_i(g_i)\chi_i(\det g_i), \quad (f \otimes \chi \circ \nu_3)(g) = f(g)\chi(\nu_3(g)),$$

there is no harm in assuming that  $\pi_i \simeq I(\alpha_F^{-t_i}, \alpha_F^{t_i})$  with  $t_i \in \mathbf{C}$  or  $\pi_i \simeq \text{St}$ . When  $\pi_i \simeq I(\alpha_F^{-t_i}, \alpha_F^{t_i})$ , we denote the unique Whittaker function which takes the value 1 on  $\text{GL}_2(\mathfrak{o})$  by  $W_i^0 \in \mathscr{W}(\pi_i)$  and let  $W_i^\pm \in \mathscr{W}(\pi_i)$  be the unique Whittaker function characterized by

$$W_i^\pm(\mathbf{t}(a)) = |a|^{(\pm 2t_i + 1)/2} \mathbb{I}_{\mathfrak{o}}(a)$$

for  $a \in F^\times$ . When  $\pi_i \simeq \text{St} \otimes \alpha_F^{s_i}$ , we define  $W_i^+ \in \mathscr{W}(\text{St} \otimes \alpha_F^{s_i})$  by

$$W_i^+(\mathbf{t}(a)) = |a|^{s_i + 1} \mathbb{I}_{\mathfrak{o}}(a)$$

and set  $t_i = s_i + \frac{1}{2}$  to be uniform. We define  $f_i^{\text{ord}} \in I(\alpha_F^{-t_i}, \alpha_F^{t_i})$  as before. Recall that  $W_i^+ = W(f_i^{\text{ord}})$ . Put

$$\eta_1 = \begin{pmatrix} 0 & -1 \\ \varpi & 0 \end{pmatrix}, \quad \mathcal{W}_i^\pm = \pi_i(\eta_1)W_i^\pm, \quad \mathcal{W}_i^0 = \pi_i(\eta_1)W_i^0.$$

**Lemma 3.1.** *If  $\pi_i$  is an irreducible unramified principal series, then*

$$W_i^0 = q^{1/2} \frac{\mathcal{W}_i^+ - \mathcal{W}_i^-}{q^{-t_i} - q^{t_i}}.$$

PROOF. The relation  $W_i^\pm = W_i^0 - q^{(\pm 2t_i - 1)/2} \mathcal{W}_i^0$  implies the stated identity in view of  $\pi_i(\eta_1)\mathcal{W}_i^0 = W_i^0$ .  $\square$

Fix an unramified character  $\chi = \alpha_F^s$  of  $F^\times$ . We will abbreviate  $I_3(\chi) = I_3(1, \chi)$ . Take  $\Phi = \mathbb{I}_{\text{Sym}_3(\mathfrak{o})}$  and put  $h^0(\chi) = f_\Phi(\chi)$ . Since

$$\mathcal{P}_3 J_3 \mathbf{n}(\text{Sym}_3(\mathfrak{o})) = \mathcal{P}_3 J_3 K_0^{(3)}(\mathfrak{p}) = \mathcal{P}_3 K_0^{(3)}(\mathfrak{p}) J_3 K_0^{(3)}(\mathfrak{p}),$$

the restriction of the section  $h^0(\chi)$  to  $\text{GSp}_6(\mathfrak{o})$  is the characteristic function of  $K_0^{(3)}(\mathfrak{p}) J_3 K_0^{(3)}(\mathfrak{p})$ . In particular,

$$\rho_3(k)h^0(\chi) = h^0(\chi)$$

for  $k \in K_0^{(3)}(\mathfrak{p})$  (cf. Lemma 2.2).

**Lemma 3.2.** *Assume that  $\pi_1 \simeq \text{St}$ . Then*

$$Z(\mathcal{W}_1^+, \mathcal{W}_2^+, \mathcal{W}_3^+, h^0(\chi)) = -\frac{q^{s-2}}{(1+q^{-1})^3} \zeta(s+1+t_2+t_3) \prod_{i=2,3} \zeta(s+1+t_i-t_{5-i}).$$

**Remark 3.3.** Lemma 3.2 is compatible with the computation [GK92]. Let  $\Phi^0(\chi) \in I_3(\chi)$  be the function whose restriction to  $\text{GSp}_6(\mathfrak{o})$  is the characteristic function of  $K_0^{(3)}(\mathfrak{p})$ . Put  $\eta_3 = \iota(\eta_1, \eta_1, \eta_1)$ . Then

$$\eta_3 K_0^{(3)}(\mathfrak{p}) \eta_3^{-1} = K_0^{(3)}(\mathfrak{p}), \quad h^0(\chi) = q^{3+3s} \rho_3(\eta_3) \Phi^0(\chi)$$

by Lemma 3.1 of [GK92]. We obtain

$$Z(\mathcal{W}_1^+, \mathcal{W}_2^+, \mathcal{W}_3^+, h^0(\chi)) = q^{3+3s} Z(W_1^+, W_2^+, W_3^+, \Phi^0).$$

When  $\pi_1 \simeq \pi_2 \simeq \pi_3 \simeq \text{St}$ , Proposition 4.2 of [GK92] gives

$$Z(W_1^+, W_2^+, W_3^+, \Phi^0) = -(q+1)^{-3} q^{-2s-2} L\left(s + \frac{1}{2}, \text{St} \times \text{St} \times \text{St}\right).$$

PROOF. On account of (2.9) we have

$$Z(\mathcal{W}_1^+, \mathcal{W}_2^+, \mathcal{W}_3^+, h^0(\chi)) = \int_{F^\times} \int_{T' \setminus \text{SL}_2(F)} \mathcal{W}_1^+(\mathbf{t}(a)g) |a|^{s+t_2+t_3} \mathcal{F}'(g) \text{d}g \text{d}^\times a.$$

Put  $f_i'' = \pi_i(\mathbf{t}(\varpi)) f_i^{\text{ord}}$ . Since

$$f_i''(J_1 \mathbf{n}(x)) = f_i^{\text{ord}}(J_1 \mathbf{t}(\varpi) \mathbf{n}(x/\varpi)) = q^{(1-2t_i)/2} \mathbb{I}_{\mathfrak{p}}(x),$$

we get

$$\begin{aligned} \mathcal{F}'(g) &= (1+q^{-1})^{-2} \int_{F^2} dx_2 dx_3 f_2''(J_1 \mathbf{n}(x_2)) f_3''(J_1 \mathbf{n}(x_3)) \int_{F^{\times 2}} \prod_{i=2,3} |a_i|^{2t_i} \frac{d^\times a_i}{|a_i|} \\ &\quad \times \int_{F^2} h^0(\iota_0(g, \mathbf{m}(a_2) \mathbf{n}^-(u_2) \mathbf{n}(x_2), \mathbf{m}(a_3) \mathbf{n}^-(u_3) \mathbf{n}(x_3)), \chi) du_2 du_3 \\ &= \int_{F^{\times 2} \oplus F^2} h^0(\iota_0(g, \mathbf{m}(a_2) \mathbf{n}^-(u_2), \mathbf{m}(a_3) \mathbf{n}^-(u_3)), \chi) \frac{\prod_{i=2,3} |a_i|^{2t_i-1} d^\times a_i du_i}{q^{1+t_2+t_3} (1+q^{-1})^2}. \end{aligned}$$

In view of (2.6)

$$\begin{aligned} \mathcal{F}'(\mathbf{n}^-(u) \mathbf{n}(x)) &= \int_{F^{\times 2} \oplus F^2} \Phi \left( \begin{pmatrix} x & -a_2 & -a_3 \\ -a_2 & -u_2 & -a_2 a_3 u \\ -a_3 & -a_2 a_3 u & -u_3 \end{pmatrix} \right) \frac{\prod_{i=2,3} |a_i|^{1+2s+2t_i} d^\times a_i du_i}{q^{1+t_2+t_3} (1+q^{-1})^2} \\ &= q^{-1-t_2-t_3} (1+q^{-1})^{-2} \mathbb{I}_0(x) \int_{\mathfrak{o}^2} \mathbb{I}_0(a_2 a_3 u) \prod_{i=2,3} |a_i|^{1+2s+2t_i} d^\times a_i. \end{aligned}$$

Owing to (2.11) we arrive at

$$\begin{aligned} &\gamma \left( s + \frac{1}{2}, \pi_1 \otimes \alpha_F^{t_2+t_3}, \psi \right) Z(\mathcal{W}_1^+, \mathcal{W}_2^+, \mathcal{W}_3^+, h^0(\chi)) \\ &= \int_{F^\times} \int_F \mathcal{W}_1^+(\mathbf{t}(a) J_1^{-1} \mathbf{n}(x)) |a|^{-s-t_2-t_3} \mathcal{F}'_\psi(a, x) dx d^\times a, \end{aligned}$$

where

$$\begin{aligned} \mathcal{F}'_\psi(a, x) &= (1+q^{-1})^{-1} \int_F \mathcal{F}'(\mathbf{n}^-(u) \mathbf{n}(x)) \psi(-au) du \\ &= q^{-1-t_2-t_3} (1+q^{-1})^{-3} \mathbb{I}_0(x) \int_{\mathfrak{o}^2} \mathbb{I}_0\left(\frac{a}{a_2 a_3}\right) \prod_{i=2,3} |a_i|^{2s+2t_i} d^\times a_i. \end{aligned}$$

We conclude that

$$\begin{aligned} &q^{1+t_2+t_3} (1+q^{-1})^3 \gamma \left( s + \frac{1}{2}, \pi_1 \otimes \alpha_F^{t_2+t_3}, \psi \right) Z(\mathcal{W}_1^+, \mathcal{W}_2^+, \mathcal{W}_3^+, h^0(\chi)) \\ &= \int_{F^\times} \int_F dx d^\times a \frac{\mathcal{W}_1^+(\mathbf{t}(a) J_1^{-1} \mathbf{n}(x))}{|a|^{s+t_2+t_3}} \mathbb{I}_0(x) \int_{\mathfrak{o}^2} \mathbb{I}_0\left(\frac{a}{a_2 a_3}\right) \prod_{i=2,3} |a_i|^{2s+2t_i} d^\times a_i \\ &= \int_{F^\times} d^\times a \frac{\mathcal{W}_1^+(\mathbf{t}(a_2 a_3 a) J_1^{-1})}{|a_2 a_3 a|^{s+t_2+t_3}} \int_{\mathfrak{o}^2} \mathbb{I}_0(a) \prod_{i=2,3} |a_i|^{2s+2t_i} d^\times a_i \\ &= \int_{\mathfrak{o}} d^\times a \frac{W_1^+(\mathbf{t}(a_2 a_3 a \varpi))}{|a|^{s+t_2+t_3}} \prod_{i=2,3} \int_{\mathfrak{o}} |a_i|^{s+t_i-t_5-i} d^\times a_i \\ &= \frac{\zeta\left(\frac{1}{2}-s+t_1-t_2-t_3\right)}{q^{(2t_1+1)/2}} \prod_{i=2,3} \zeta\left(s+\frac{1}{2}+t_1+t_i-t_5-i\right). \end{aligned}$$

Assume that  $\pi_1 \simeq \text{St}$ . Then  $t_1 = \frac{1}{2}$  and

$$\gamma \left( s + \frac{1}{2}, \pi_1 \otimes \alpha_F^{t_2+t_3}, \psi \right) = -q^{-s-t_2-t_3} \frac{\zeta(1-s-t_2-t_3)}{\zeta(s+1+t_2+t_3)},$$

from which we complete our proof.  $\square$

**Proposition 3.4.** *Let  $\pi_i$  be either an unramified principal series representation or the Steinberg representation twisted by an unramified character. Set  $W_i = W_i^0$  in the former case and  $W_i = W_i^+$  in the latter case. Put*



$\check{W}_i = W_i \otimes \omega_i^{-1}$ . If not all  $\pi_i$  are principal series, then for an unramified character  $\chi$  of  $F^\times$

$$\begin{aligned} Z(W_1, W_2, W_3, h^0(\chi)) &= Z(\check{W}_1, \check{W}_2, \check{W}_3, h^0(\chi\hat{\omega})) \\ &= -(\hat{\omega}^2\chi^4)(\varpi)q(1+q)^{-3} \frac{L\left(\frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi\right)}{\varepsilon\left(\frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi, \psi\right)}. \end{aligned}$$

**Remark 3.5.** If  $\pi_1$  and  $\pi_2$  are irreducible unramified principal series representations, then

$$\begin{aligned} L(s, \pi_1 \times \pi_2 \times \text{St}) &= L\left(s + \frac{1}{2}, \pi_1 \times \pi_2\right), & \varepsilon(s, \pi_1 \times \pi_2 \times \text{St}, \psi) &= q^{-4s+2}\omega_1(\varpi)^2\omega_2(\varpi)^2, \\ L(s, \pi_1 \times \text{St} \times \text{St}) &= L(s, \pi_1)L(s+1, \pi_1), & \varepsilon(s, \pi_1 \times \text{St} \times \text{St}, \psi) &= q^{-4s+2}\omega_1(\varpi)^2, \\ L(s, \text{St} \times \text{St} \times \text{St}) &= \zeta\left(s + \frac{3}{2}\right)\zeta\left(s + \frac{1}{2}\right)^2, & \varepsilon(s, \text{St} \times \text{St} \times \text{St}, \psi) &= -q^{-(10s-5)/2}. \end{aligned}$$

PROOF. In view of [Ike89, Lemma 3.1] and (3.1) we may assume that  $\pi_1 \simeq \text{St}$  and  $\pi_i$  is a quotient of  $I(\alpha_F^{-t_i}, \alpha_F^{t_i})$  for  $i = 2, 3$ . If all  $\pi_i$  are discrete series representations, then since  $\mathcal{W}_1 = -W_1$ , the result follows from Lemma 3.2. Let  $\chi = \alpha_F^s$  and  $\pi_3 \simeq I(\alpha_F^{-t_3}, \alpha_F^{t_3})$ . Lemma 3.2 gives

$$Z(W_1^+, \mathcal{W}_2^+, \mathcal{W}_3^\pm, h^0(\chi)) = \frac{q^{s-2}}{(1+q^{-1})^3} L(s+1+t_2, \pi_3)\zeta(s+1-t_2 \pm t_3).$$

Thanks to Lemma 3.1 we obtain

$$\begin{aligned} \frac{Z(W_1^+, \mathcal{W}_2^+, W_3^0, h^0(\chi))}{q^{s-2}L(s+1+t_2, \pi_3)} &= q^{1/2} \frac{\zeta(s+1-t_2+t_3) - \zeta(s+1-t_2-t_3)}{(1+q^{-1})^3(q^{-t_3} - q^{t_3})} \\ &= (1+q^{-1})^{-3} q^{1/2} q^{-s-1+t_2} L(s+1-t_2, \pi_3). \end{aligned}$$

If  $\pi_2 \simeq I(\alpha_F^{-t_2}, \alpha_F^{t_2})$ , then

$$Z(W_1^+, \mathcal{W}_2^+, W_3^0, h^0(\chi)) = (1+q^{-1})^{-3} q^{(2t_2-5)/2} L(s+1, \pi_2 \times \pi_3),$$

and so again by Lemma 3.1,

$$Z(W_1^+, W_2^0, W_3^0, h^0(\chi)) = (1+q^{-1})^{-3} L(s+1, \pi_2 \times \pi_3) \frac{q^{t_2-2} - q^{-t_2-2}}{q^{-t_2} - q^{t_2}} = -(1+q^{-1})^{-3} q^{-2} L(s+1, \pi_2 \times \pi_3).$$

If  $\pi_2 \simeq \text{St}$ , we obtain the claimed result by letting  $t_2 = \frac{1}{2}$ .  $\square$

#### 4. COMPUTATION OF THE LOCAL ZETA INTEGRAL: THE ARCHIMEDEAN CASE

**4.1. Archimedean sections.** We define the sign character  $\text{sgn} : \mathbf{R}^\times \rightarrow \{\pm 1\}$  by  $\text{sgn}(x) = \frac{x}{|x|}$ . Let  $\text{Sym}_n^+(\mathbf{R})$  denote the set of positive definite symmetric matrices of rank  $n$ . The Siegel upper half-space  $\mathfrak{H}_n$  of degree  $n$  consists of complex symmetric matrices of size  $n$  with positive definite imaginary part. The Lie group  $\text{GSp}_n^+(\mathbf{R}) = \{g \in \text{GSp}_n(\mathbf{R}) \mid \nu_n(g) > 0\}$  acts on the space  $\mathfrak{H}_n$  by  $gZ = (AZ + B)(CZ + D)^{-1}$ , where  $Z \in \mathfrak{H}_n$  and  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  with matrices  $A, B, C, D$  of size  $n$ . Let  $C^\infty(\mathfrak{H}_n)$  be the space of  $\mathbf{C}$ -valued smooth functions on the upper half complex plane  $\mathfrak{H}_n$ . For an integer  $k$  and  $f \in C^\infty(\mathfrak{H}_n)$  we define

$$(4.1) \quad f|_k g(Z) = f(gZ)J(g, Z)^{-k}, \quad J(g, Z) = \nu_n(g)^{-n/2} \det(CZ + D).$$

Put  $\mathbf{i} = \sqrt{-1}\mathbf{1}_n$ . We will identify the compact unitary group  $U(n) = \{u \in \text{GL}_n(\mathbf{C}) \mid \bar{u}^t u = \mathbf{1}_n\}$  with the fixator  $\{g \in \text{Sp}_n(\mathbf{R}) \mid g(\mathbf{i}) = \mathbf{i}\}$  via the map  $g \mapsto \bar{J}(g, \mathbf{i})$ .

For  $1 \leq i, j \leq 3$  and  $u \in U(3)$  we define  $H_{ij}(u)$  to be the  $(i, j)$ -entry of the matrix  $u^t u$ . By definition,  $H_{ij}$  is a function on  $O(3) \backslash U(3)$ , and hence we can extend it to a unique function on  $\text{GSp}_6(\mathbf{R})$  such that

$$H_{ij}(\mathbf{n}(z)\mathbf{m}(A, \nu)u) = H_{ij}(u) \quad (z \in \text{Sym}_3(\mathbf{R}), A \in \text{GL}_3(\mathbf{R}), \nu \in \mathbf{R}^\times, u \in U(3)).$$

A *parity type* is a triplet  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$  of integers which belongs to one of the following triplets

$$\lambda \in \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 2)\}.$$

Fix a parity type  $\lambda$  and a character  $\chi_\infty$  of  $\mathbf{R}^\times$ . Put

$$H_\lambda := \begin{cases} 1 & \text{if } \lambda = (0, 0, 0), \\ \overline{H_{23}} & \text{if } \lambda = (0, 1, 1), \\ H_{12} & \text{if } \lambda = (1, 0, 1), \\ H_{12}\overline{H_{23}} & \text{if } \lambda = (1, 1, 2). \end{cases}$$

For each integer  $k$  we define  $f_{\infty, s}^{[k, \lambda]} \in I_3(\text{sgn}^{k-\lambda_1}, \chi_\infty \text{sgn}^{k-\lambda_1} \alpha_{\mathbf{R}}^s)$  by

$$f_{s, \infty}^{[k, \lambda]}(g) := H_\lambda(g) \chi_\infty(\nu_3(g)) \cdot J(g, \mathbf{i})^{-k+\lambda_1} |J(g, \mathbf{i})|^{k-\lambda_1-2s-2}.$$

Since

$$H_{ij}(g\iota(\kappa_{\theta_1}, \kappa_{\theta_2}, \kappa_{\theta_3})) = e^{\sqrt{-1}(\theta_i + \theta_j)} H_{ij}(g), \quad \kappa_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},$$

we have

$$(4.2) \quad f_{s, \infty}^{[k, \lambda]}(g\iota(\kappa_{\theta_1}, \kappa_{\theta_2}, \kappa_{\theta_3})) = f_{s, \infty}^{[k, \lambda]}(g) e^{\sqrt{-1}\{k\theta_1 + (k-\lambda_2)\theta_2 + (k-\lambda_3)\theta_3\}}.$$

**4.2. Archimedean degenerate Whittaker functions.** For a positive integer  $m$  we put

$$\Gamma_m(s) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma\left(s - \frac{j}{2}\right).$$

If  $h$  is positive definite and  $\alpha, \beta \in \mathbf{C}$ , then the integral

$$\omega(h; \alpha, \beta) = \frac{\det(h)^\beta}{\Gamma_3(\beta)} \int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(uh)} \det(u + \mathbf{1}_3)^{\alpha-2} (\det u)^{\beta-2} du$$

is absolutely convergent for  $\text{Re } \beta > 2$  and can be continued to a holomorphic function on  $\mathbf{C} \times \mathbf{C}$  by Theorem 3.1 of [Shi82]. It is convenient to introduce the function  $\omega^*(h; \alpha, \beta)$  given by

$$(4.3) \quad \begin{aligned} \omega^*(h; \alpha, \beta) &:= \det(4\pi h)^{\alpha-2} \cdot \omega(4\pi h; \alpha, \beta) \\ &= \frac{1}{\Gamma_3(\beta)} \int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(u)} \det(u + 4\pi h)^{\alpha-2} (\det u)^{\beta-2} du. \end{aligned}$$

It follows from this expression that if  $\alpha \in \mathbf{Z}$  and  $\alpha \geq 2$ , then  $\omega^*(h; \alpha, \beta)$  is a polynomial function in  $h$  of degree at most  $\alpha - 2$  and makes sense for an arbitrary symmetric matrix  $h$ .

**Lemma 4.1.** *For  $x \in \text{Sym}_3(\mathbf{R})$  we have*

$$\begin{aligned} H_{23}(J_3 \mathbf{n}(x)) &= 2\sqrt{-1}(x_{11}x_{23} - x_{12}x_{13} + \sqrt{-1}x_{23}) / \det(x + \mathbf{i}), \\ H_{12}(J_3 \mathbf{n}(x)) &= 2\sqrt{-1}(x_{12}x_{33} - x_{23}x_{13} + \sqrt{-1}x_{12}) / \det(x + \mathbf{i}). \end{aligned}$$

**PROOF.** The Iwasawa decomposition of  $J_3 \mathbf{n}(x)$  can be written as

$$J_3 \mathbf{n}(x) = \begin{pmatrix} z^\dagger & * \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} zx & -z \\ z & zx \end{pmatrix}, \quad z \in \text{GL}_3(\mathbf{R})$$

with  $z^\dagger z = (\mathbf{1}_3 + x^2)^{-1}$ . Let  $u = z(x - \mathbf{i}) \in \text{U}(3)$ . Then  $u^\dagger u = (x - \mathbf{i})(x + \mathbf{i})^{-1}$ . We denote the adjugate of a matrix  $A \in \text{M}_3(\mathbf{R})$  by  $\text{adj}(A)$ . Since  $A \cdot \text{adj}(A) = (\det A)\mathbf{1}_3$ , we have

$$u^\dagger u = \det(x + \mathbf{i})^{-1} (x - \mathbf{i}) \text{adj}(x + \mathbf{i}) = -2\sqrt{-1} \det(x + \mathbf{i})^{-1} \text{adj}(x + \mathbf{i}) + \mathbf{1}_3.$$

By definition we find that

$$\begin{aligned} H_{23}(J_3 \mathbf{n}(x)) &= H_{23}(u) = \det(x + \mathbf{i})^{-1} \cdot 2\sqrt{-1} \det \begin{pmatrix} x_{11} + \sqrt{-1} & x_{12} \\ x_{13} & x_{23} \end{pmatrix} \\ &= \det(x + \mathbf{i})^{-1} \cdot 2\sqrt{-1}(x_{11}x_{23} - x_{12}x_{13} + \sqrt{-1}x_{23}). \end{aligned}$$

One can compute  $H_{12}(J_3 \mathbf{n}(x))$  in the same way. □

**Definition 4.2.** We associate to a parity type  $\lambda$  the differential operator  $\mathcal{D}_\lambda$  on  $T = (T_{ij}) \in \text{Sym}_3(\mathbf{R})$  by

$$\begin{aligned}\mathcal{D}_{(0,1,1)} &:= \frac{1}{2\pi^2\sqrt{-1}} \{\partial_{13}\partial_{12} - \partial_{23}(\partial_{11} - 4\pi)\}, & \mathcal{D}_{(0,0,0)} &= \text{id}, \\ \mathcal{D}_{(1,0,1)} &:= \frac{1}{2\pi^2\sqrt{-1}} \{\partial_{12}\partial_{33} - \partial_{23}\partial_{13}\}, & \mathcal{D}_{(1,1,2)} &:= \mathcal{D}_{(0,1,1)}\mathcal{D}_{(1,0,1)}.\end{aligned}$$

Here

$$\partial_{ij} := \frac{\partial}{\partial T_{ij}} \cdot \begin{cases} 1 & \text{if } i = j, \\ \frac{1}{2} & \text{if } i \neq j. \end{cases}$$

**Definition 4.3.** For each parity type  $\lambda$  and an integer  $\lambda_2 \leq r \leq k-2$  we put  $M = k-r-2$  and define

$$\begin{aligned}\mathbf{K}_{\mathcal{D}_\lambda}^M(T; u) &:= \mathcal{D}_\lambda \{\det(4\pi T + u)^M\}, \\ \omega_{\mathcal{D}_\lambda}^M(T, s) &:= \frac{1}{\Gamma_3(s)} \int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(u)} \mathbf{K}_{\mathcal{D}_\lambda}^M(T; u) (\det u)^{s-2} du = \mathcal{D}_\lambda \omega^*(T; M+2, s).\end{aligned}$$

**Lemma 4.4.** Let  $A \in \text{GL}_3(\mathbf{R})^+$  and  $B \in \text{Sym}_3(\mathbf{Q})$  with  $\det B \neq 0$ . If  $B$  is positive definite, then

$$\lim_{s \rightarrow \frac{k-\lambda_1}{2} - r - 1} \mathcal{W}_B(\mathbf{m}(A), f_{s, \infty}^{[k, \lambda]}) = C_1^{[k, r, \lambda]} e^{-2\pi \text{tr}(A^t B A)} \frac{\omega_{\mathcal{D}_\lambda}^M(A^t B A; \lambda_2 - r)}{(\det A)^{k-\lambda_1-2r-4}},$$

where

$$C_1^{[k, r, \lambda]} = (\sqrt{-1})^{k-\lambda_2} \frac{2^{3(3+2r-k-\lambda_2)} \pi^6}{\Gamma_3(k-r)}.$$

If  $B$  is not positive definite, then for any integer  $0 \leq r < k-1$ ,

$$\lim_{s \rightarrow \frac{k-\lambda_1}{2} - r - 1} \mathcal{W}_B(\mathbf{m}(A), f_{s, \infty}^{[k, \lambda]}) = 0.$$

**PROOF.** For each parity type  $\lambda$  we define another differential operator  $\mathcal{D}_\lambda$  on  $\text{Sym}_3(\mathbf{R})$  by

$$\begin{aligned}\mathcal{D}_{(0,1,1)} &:= \frac{1}{2\pi^2\sqrt{-1}} \{\partial_{13}\partial_{12} - \partial_{23}(\partial_{11} - 2\pi)\}, & \mathcal{D}_{(0,0,0)} &:= \text{id}, \\ \mathcal{D}_{(1,0,1)} &:= \frac{1}{2\pi^2\sqrt{-1}} \{\partial_{12}(\partial_{33} + 2\pi) - \partial_{23}\partial_{13}\}, & \mathcal{D}_{(1,1,2)} &:= \mathcal{D}_{(0,1,1)}\mathcal{D}_{(1,0,1)}.\end{aligned}$$

It should be remarked that by Lemma 4.1

$$\mathcal{D}_\lambda(e^{-2\pi\sqrt{-1}\text{tr}(Tx)}) = \det(x + \mathbf{i})^{\lambda_1} \det(x - \mathbf{i})^{\lambda_2} H_\lambda(J_3 \mathbf{n}(x)) e^{-2\pi\sqrt{-1}\text{tr}(Tx)}.$$

Recall that

$$\mathcal{W}_B(\mathbf{m}(A), f_{s, \infty}^{[k, \lambda]}) = (\det A)^{-2s-2} \mathcal{W}_{A^t B A}(\mathbf{1}_6, f_{s, \infty}^{[k, \lambda]}),$$

which reduces our computation to the case  $A = \mathbf{1}_3$ . We see that

$$\begin{aligned}\mathcal{W}_B(\mathbf{1}_6, f_{s, \infty}^{[k, \lambda]}) &= \int_{\text{Sym}_3(\mathbf{R})} \det(x + \mathbf{i})^{-\alpha_0} \det(x - \mathbf{i})^{-\beta_0} H_\lambda(J_3 \mathbf{n}(x)) e^{-2\pi\sqrt{-1}\text{tr}(Bx)} dx \\ &= \mathcal{D}_\lambda(\xi(\mathbf{1}_3, T; \alpha_0 + \lambda_1, \beta_0 + \lambda_2))|_{T=B}\end{aligned}$$

with  $\alpha_0 = s + 1 + \frac{k-\lambda_1}{2}$  and  $\beta_0 = s + 1 - \frac{k-\lambda_1}{2}$ . On the other hand, for any  $h \in \text{Sym}_3(\mathbf{R})$ , we have

$$\xi(\mathbf{1}_3, h; \alpha, \beta) = (\sqrt{-1})^{3(\beta-\alpha)} \frac{(2\pi)^6 e^{-2\pi\text{tr}(h)}}{2^3 \Gamma_3(\alpha) \Gamma_3(\beta)} \int_{u>0, u>-2\pi h} e^{-2\text{tr}(u)} \det(u + 2\pi h)^{\alpha-2} (\det u)^{\beta-2} du$$

by [Shi82, (1.29)]. If  $h$  is positive definite, then the last integral equals  $2^{3(2-\alpha-\beta)} \omega^*(h; \alpha, \beta) \cdot \Gamma_3(\beta)$ . Observe that for every polynomial  $P$  on  $\text{Sym}_3(\mathbf{R})$

$$\mathcal{D}_\lambda(e^{-2\pi\text{tr}(T)} P(T)) = e^{-2\pi\text{tr}(T)} \mathcal{D}_\lambda P(T).$$

This proves the case where  $B$  is positive definite. If the signature of  $B$  is  $(3-q, q)$ , then Theorem 4.2 of [Shi82] gives that a holomorphic function  $\tilde{\omega}(\alpha, \beta)$  such that

$$\xi(\mathbf{1}_3, B; \alpha, \beta) = \frac{\Gamma_p(\beta - \frac{q}{2}) \Gamma_q(\alpha - \frac{p}{2})}{\Gamma_3(\alpha) \Gamma_3(\beta)} \cdot \tilde{\omega}(\alpha, \beta).$$

Thus  $\xi(\mathbf{1}_3, B; k-r, -r) = 0$  for  $0 \leq r \leq k-2$  unless  $B$  is positive definite.  $\square$

**4.3. The constant term of  $\mathcal{W}_B(\mathbf{m}(\text{diag}(\sqrt{y_1}, \sqrt{y_2}, \sqrt{y_3})), f_{s, \infty}^{[k, \lambda]})$  as a polynomial of  $y_1^{-1}$ .** Given  $y = \text{diag}(y_1, y_2, y_3) \in \mathbf{R}_+^3$ , we put  $A = \text{diag}(\sqrt{y_1}, \sqrt{y_2}, \sqrt{y_3})$  and define

$$\mathbf{W}_B^{[k, r, \lambda]}(y) := (y_1 y_2 y_3)^{r-k+2} \sqrt{y_1}^{\lambda_1} \sqrt{y_2}^{\lambda_1 + \lambda_2} \sqrt{y_3}^{2\lambda_1 + \lambda_2} \cdot \omega_{\mathcal{D}_\lambda}^M(A^t B A, \lambda_2 - r).$$

Now we write

$$\omega^*(T; M+2, s) = \sum_{0 \leq j_1, j_2, j_3 \leq M} c_{j_1 j_2 j_3} T_{12}^{j_3} T_{23}^{j_1} T_{13}^{j_2}, \quad c_{j_1 j_2 j_3} \in \mathbf{C}[T_{11}, T_{22}, T_{33}],$$

where  $T = (T_{ij}) \in \text{Sym}_3^+(\mathbf{R})$ . Since

$$\omega^*(\varepsilon^t T \varepsilon; M+2, s) = \omega^*(T; M+2, s), \quad \varepsilon = \text{diag}(-1, 1, 1)$$

in view of the expression (4.3), we get  $c_{j_1 j_2 j_3} = (-1)^{j_2 + j_3} c_{j_1 j_2 j_3}$ . Thus  $c_{j_1 j_2 j_3} = 0$  unless  $j_2 \equiv j_3 \pmod{2}$ . By symmetry we conclude that  $c_{j_1 j_2 j_3} = 0$  unless  $j_1 \equiv j_2 \equiv j_3 \pmod{2}$ . Moreover, we can write

$$T_{23}^{\lambda_1} T_{12}^{\lambda_2} T_{13}^{\lambda_1 + \lambda_2} \omega_{\mathcal{D}_\lambda}^*(T, s) = \sum_{0 \leq j_1, j_2, j_3 \leq M, j_1 \equiv j_2 \equiv j_3 \pmod{2}} a_{j_1 j_2 j_3} T_{12}^{j_3} T_{23}^{j_1} T_{13}^{j_2}, \quad a_{j_1 j_2 j_3} \in \mathbf{C}[T_{11}, T_{22}, T_{33}].$$

Thus we can write

$$(4.4) \quad \mathbf{W}_B^{[k, r, \lambda]}(\text{diag}(y_1, y_2, y_3)) = \sum_{0 \leq a, b, c \leq M} Q_{a, b, c}^{[k, \lambda]}(B, r) y_1^{-a} y_2^{-b} y_3^{-c}.$$

We shall determine the coefficient  $Q_{0, b, c}^{[k, \lambda]}(B, r)$  of  $\mathbf{W}_B^{[k, r, \lambda]}(y)$  for matrices  $B$  with zero diagonal entries

$$B = \begin{pmatrix} 0 & b_3 & b_2 \\ b_3 & 0 & b_1 \\ b_2 & b_1 & 0 \end{pmatrix}.$$

Let  $\mathcal{Y}$  be the matrix with variables  $Y_1, Y_2, Y_3$  given by

$$\mathcal{Y} = \begin{pmatrix} 0 & \sqrt{Y_1 Y_2} & \sqrt{Y_1 Y_3} \\ \sqrt{Y_1 Y_2} & 0 & \sqrt{Y_2 Y_3} \\ \sqrt{Y_1 Y_3} & \sqrt{Y_2 Y_3} & 0 \end{pmatrix}.$$

For two functions  $f, g : \mathbf{R}_+ \rightarrow \mathbf{C}$  and  $c \in \mathbf{R}$  we say that  $f(y) = g(y) + o(y^c)$  if  $\lim_{y \rightarrow \infty} \frac{f(y) - g(y)}{y^c} = 0$ .

**Lemma 4.5.** *The polynomial  $\mathbf{K}_{\mathcal{D}_\lambda}^M(\frac{\mathcal{Y}}{4\pi}; u) \in \mathbf{C}[\sqrt{Y_1}, \sqrt{Y_2}, \sqrt{Y_3}, u]$  in Definition 4.3 has the form*

$$\mathbf{K}_{\mathcal{D}_\lambda}^M((4\pi)^{-1} \mathcal{Y}; u) = C_2^{[k, r, \lambda]} \mathbf{c}_\lambda(Y_2, Y_3; u) \cdot Y_1^{M - \frac{\lambda_1}{2}} + o(Y_1^{M - \frac{\lambda_1}{2}})$$

with  $C_2^{[k, r, \lambda]} \in \mathbf{C}$  and  $\mathbf{c}_\lambda(Y_2, Y_3; u) \in \mathbf{C}[\sqrt{Y_2}, \sqrt{Y_3}, u]$  give by

$$C_2^{[k, r, \lambda]} = \frac{(2M + \lambda_1)!}{(2M)!} \cdot \frac{2^{3(\lambda_1 + \lambda_2) - \lambda_1} M!}{(\sqrt{-1})^{\lambda_2 - \lambda_1} (M - \lambda_1 - \lambda_2)!},$$

$$\mathbf{c}_\lambda(Y_2, Y_3; u) = \left( -u_{22} Y_3 - u_{33} Y_2 + 2Y_2 Y_3 + 2u_{23} \sqrt{Y_2 Y_3} \right)^{M - \lambda_1 - \lambda_2} \cdot \sqrt{Y_2}^{\lambda_1 + \lambda_2} \sqrt{Y_3}^{\lambda_2}.$$

**PROOF.** This is proved by a direct computation. Note that

$$\begin{aligned} \partial_{11} \det(T + u) &= (T_{22} + u_{22})(T_{33} + u_{33}) - (T_{23} + u_{23})^2, \\ \partial_{12} \det(T + u) &= -(T_{12} + u_{12})(T_{33} + u_{33}) + (T_{23} + u_{23})(T_{13} + u_{13}), \\ \partial_{13} \det(T + u) &= -(T_{13} + u_{13})(T_{22} + u_{22}) + (T_{12} + u_{12})(T_{23} + u_{23}), \\ \partial_{23} \det(T + u) &= -(T_{23} + u_{23})(T_{11} + u_{11}) + (T_{12} + u_{12})(T_{13} + u_{13}), \\ \partial_{33} \det(T + u) &= (T_{11} + u_{11})(T_{22} + u_{22}) - (T_{12} + u_{12})^2. \end{aligned}$$

Put

$$\Delta = \det(T + u), \quad R = (-u_{22} Y_3 - u_{33} Y_2 + 2Y_2 Y_3 + 2u_{23} \sqrt{Y_2 Y_3}) Y_1.$$

Since  $\Delta|_{T=y} = R + o(Y_1)$ , we have

$$\begin{aligned} & \mathbf{K}_{\mathcal{D}_{(0,1,1)}}^M((4\pi)^{-1}\mathcal{Y}; u) \\ &= (2\pi^2\sqrt{-1})^{-1} \cdot (4\pi)^2 \{\partial_{13}\partial_{12} - \partial_{23}(\partial_{11} - 1)\} \Delta^M|_{T=y} \\ &\equiv -8\sqrt{-1} [M(M-1)R^{M-2}(\partial_{13}\Delta\partial_{12}\Delta - \partial_{23}\Delta\partial_{11}\Delta) + MR^{M-1}\{\partial_{13}\partial_{12} - \partial_{23}(\partial_{11} - 1)\}\Delta]|_{T=y} \\ &\equiv -8\sqrt{-1} MR^{M-1} \partial_{23}\Delta|_{T=y} \\ &\equiv -8\sqrt{-1} MR^{M-1} \sqrt{Y_2 Y_3} Y_1 \pmod{o(Y_1^M)}, \end{aligned}$$

which verifies the case  $\lambda = (0, 1, 1)$ . When  $\lambda = (1, 0, 1)$ , we have

$$\begin{aligned} & \mathbf{K}_{\mathcal{D}_{(1,0,1)}}^M((4\pi)^{-1}\mathcal{Y}; u) \\ &\equiv -8\sqrt{-1} \{M(M-1)R^{M-2}(\partial_{12}\Delta\partial_{33}\Delta - \partial_{13}\Delta\partial_{23}\Delta) + MR^{M-1}(\partial_{12}\partial_{33}\Delta - \partial_{13}\partial_{23}\Delta)\}|_{T=y} \\ &\equiv -8\sqrt{-1} \left\{ M(M-1)R^{M-2}(-R\sqrt{Y_1 Y_2}) + MR^{M-1} \left( -\frac{3}{2}\sqrt{Y_1 Y_2} \right) \right\} \\ &\equiv 4\sqrt{-1} M(2M+1)R^{M-1} \sqrt{Y_1 Y_2} \pmod{o(Y_1^{M-\frac{1}{2}})} \end{aligned}$$

as claimed. Since

$$\mathcal{D}_{(0,1,1)}\Delta^M|_{T=y} = -8\sqrt{-1} M\Delta^{M-1} T_{12} T_{13}|_{T=y} + o(Y_1^M),$$

we have

$$\begin{aligned} \mathbf{K}_{\mathcal{D}_{(1,1,2)}}^M((4\pi)^{-1}\mathcal{Y}; u) &\equiv 32M(M-1)(2M-1)R^{M-2} \sqrt{Y_1 Y_2} Y_1 \sqrt{Y_2 Y_3} \\ &\quad - 64M(M-1)\Delta^{M-2} (T_{13}\partial_{33}\Delta - T_{12}\partial_{23}\Delta)|_{T=y} \pmod{o(Y_1^{M-\frac{1}{2}})}, \end{aligned}$$

which proves the case  $\lambda = (1, 1, 2)$ . □

**Lemma 4.6.** *Let  $F(T)$  be a polynomial in  $T = (T_{ij}) \in \text{Sym}_3(\mathbf{R})$ . Then we have*

$$\int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(u)} F(u) \frac{(\det u)^{s-2}}{\Gamma_3(s)} du = F(-\partial_{ij})(\det T)^{-s}|_{T=\mathbf{1}_3}.$$

PROOF. If  $T$  is positive definite and  $\text{Re } s > 2$ , then

$$\int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(Tu)} \frac{(\det u)^{s-2}}{\Gamma_3(s)} du = (\det T)^{-s}$$

by [Shi81, (1.14)]. The declared formula follows immediately from the fact that

$$F(-\partial_{ij})(e^{-\text{tr}(Tu)}) = F(u)e^{-\text{tr}(Tu)}. \quad \square$$

Now let  $k \geq l \geq m$  be a set of balanced integers. We say that  $(k, l, m)$  has the parity type  $\lambda$  if

$$\lambda_1, \lambda_2 \in \{0, 1\}, \quad \lambda_1 \equiv l - m \pmod{2}, \quad \lambda_2 \equiv k - l \pmod{2}, \quad \lambda_3 = \lambda_1 + \lambda_2.$$

**Lemma 4.7.** *Let  $\lambda$  be the parity type of  $k \geq l \geq m$  and  $r$  an integer such that  $k - \frac{l+m+\lambda_1}{2} \leq r \leq \frac{l+m}{2} - 2$ . Put*

$$M = k - r - 2, \quad b = \frac{1}{2}(k - l - \lambda_2), \quad c = \frac{1}{2}(k - m - \lambda_3), \quad n = M + \frac{1}{2}(l + m - \lambda_1).$$

Then we have

$$Q_{0,b,c}^{[k,\lambda]}(B, r) = w_{0,b,c} \cdot (b_1 b_2 b_3)^n b_1^{-k} b_2^{-l} b_3^{-m},$$

where

$$w_{0,b,c} = (4\pi)^{3M-b-c-2\lambda_1-\lambda_2} 2^{M+\lambda_1+2\lambda_2-b-c} \frac{(\sqrt{-1})^{\lambda_1-\lambda_2} (2M+\lambda_1)! M!}{(2M)!(M-\lambda_1-\lambda_2-b-c)! b! c! (r-\lambda_2-b-c)!} \frac{(r-\lambda_2)!}{(r-\lambda_2-b-c)!}.$$

PROOF. Substitute  $Y_i = 4\pi \frac{b_1 b_2 b_3}{b_i^2} y_i$  into the matrix  $\mathcal{Y}$ . Then  $\mathcal{Y} = 4\pi A^t B A$  and

$$\mathbf{W}_B^{[k,r,\lambda]}(y) = \sum_{a,b,c} Q_{a,b,c}^{[k,\lambda]}(B,r) (4\pi)^{a+b+c} Y_1^{-a} Y_2^{-b} Y_3^{-c} \cdot b_1^{b+c-a} b_2^{a+c-b} b_3^{a+b-c}.$$

On the other hand,

$$\mathbf{W}_B^{[k,r,\lambda]}(y) = \left( \frac{(4\pi)^3 b_1 b_2 b_3}{Y_1 Y_2 Y_3} \right)^M \frac{\sqrt{Y_1}^{\lambda_1} \sqrt{Y_2}^{\lambda_1+\lambda_2} \sqrt{Y_3}^{2\lambda_1+\lambda_2}}{(4\pi)^{2\lambda_1+\lambda_2} b_1^{\lambda_1+\lambda_2} b_2^{\lambda_1}} \omega_{\mathcal{Y}}^*((4\pi)^{-1} \mathcal{Y}, \lambda_2 - r)$$

by definition. The equations above give a complex number  $w_{0,b,c}$  such that

$$Q_{0,b,c}^{[k,\lambda]}(B,r) = w_{0,b,c} \cdot (b_1 b_2 b_3)^n b_1^{-k} b_2^{-l} b_3^{-m}.$$

Our task is to determine  $w_{0,b,c}$ . It is the coefficient of  $Y_2^{M-\lambda_1-\lambda_2-b} Y_3^{M-\lambda_1-\lambda_2-c}$  in the polynomial

$$\begin{aligned} & \frac{(4\pi)^{3M-b-c-2\lambda_1-\lambda_2}}{\sqrt{Y_2}^{\lambda_1+\lambda_2} \sqrt{Y_3}^{\lambda_2}} \int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(u)} C_2^{[k,r,\lambda]} \mathbf{c}_\lambda(Y_2, Y_3; u) \frac{(\det u)^{s-2}}{\Gamma_3(s)} du \Big|_{s=\lambda_2-r} \\ &= (4\pi)^{3M-b-c-2\lambda_1-\lambda_2} C_2^{[k,r,\lambda]} \\ & \quad \times \int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(u)} \left( -u_{22} Y_3 - u_{33} Y_2 + 2Y_2 Y_3 + 2u_{23} \sqrt{Y_2 Y_3} \right)^{M-\lambda_1-\lambda_2} \frac{(\det u)^{s-2}}{\Gamma_3(s)} du \Big|_{s=\lambda_2-r} \end{aligned}$$

by Lemma 4.5. Put

$$L = M - \lambda_1 - \lambda_2, \quad r_1 = r - \lambda_2.$$

Notice that  $b \leq c$  by assumption. The coefficient of  $Y_2^{L-b} Y_3^{L-c}$  in the last integral is given by

$$\begin{aligned} & \sum_{i=0}^b \frac{2^{L-b-c} (-1)^{b+c} \cdot L!}{(b-i)! (c-i)! (L-b-c)! (2i)!} \int_{\text{Sym}_3^+(\mathbf{R})} e^{-\text{tr}(u)} u_{33}^{b-i} u_{22}^{c-i} (2u_{23})^{2i} \frac{(\det u)^{s-2}}{\Gamma_3(s)} du \Big|_{s=r_1} \\ &= \sum_{i=0}^b \frac{2^{L-b-c} \cdot L! 2^{2i}}{(b-i)! (c-i)! (L-b-c)! (2i)!} \partial_{33}^{b-i} \partial_{22}^{c-i} \partial_{23}^{2i} (T_{22} T_{33} - T_{23}^2)^{r_1} \Big|_{T_{22}=T_{33}=1, T_{23}=0} \\ &= \sum_{i=0}^b \frac{2^{L-b-c} \cdot L! 2^{2i}}{(b-i)! (c-i)! (L-b-c)! (2i)!} \sum_{j=0}^{r_1} \binom{r_1}{j} \partial_{33}^{b-i} \partial_{22}^{c-i} \partial_{23}^{2i} T_{22}^{r_1-j} T_{33}^{r_1-j} (-T_{23}^2)^j \Big|_{T_{22}=T_{33}=1, T_{23}=0} \\ &= \sum_{i=0}^b \frac{2^{L-b-c} \cdot L!}{(b-i)! (c-i)! (L-b-c)!} \binom{r_1}{i} (-1)^i \partial_{33}^{b-i} \partial_{22}^{c-i} (T_{22}^{r_1-i} T_{33}^{r_1-i}) \Big|_{T_{22}=T_{33}=1} \\ &= \frac{2^{L-b-c} \cdot L!}{(L-b-c)!} \sum_{i=0}^b \binom{r_1}{i} \binom{r_1-i}{b-i} \binom{r_1-i}{c-i} (-1)^i \end{aligned}$$

in view of Lemma 4.6. The last summation equals

$$\frac{r_1!}{(r_1-b)! b!} \sum_{i=0}^b \binom{b}{i} \binom{r_1-i}{r_1-c} (-1)^i = \frac{r_1!}{(r_1-b)! b!} \cdot \binom{r_1-b}{r_1-b-c} = \frac{r_1!}{b! c! (r_1-b-c)!},$$

where we can deduce this equality by equating the terms of degree  $r_1 - c$  of the identity

$$\sum_{i=0}^b \binom{b}{i} (1+X)^{r_1-i} (-1)^i = (1+X)^{r_1} \left( 1 - \frac{1}{1+X} \right)^b = (1+X)^{r_1-b} X^b,$$

Finally, we see that  $w_{0,b,c}$  equals

$$(4\pi)^{3M-b-c-2\lambda_1-\lambda_2} C_2^{[k,r,\lambda]} \frac{2^{M-\lambda_1-\lambda_2-b-c} \cdot (M-\lambda_1-\lambda_2)!}{(M-\lambda_1-\lambda_2-b-c)!} \frac{(r-\lambda_2)!}{b! c! (r-\lambda_2-b-c)!}$$

by putting together the above computations, which completes our proof.  $\square$

4.4. **The archimedean zeta integral.** Let  $V_{\pm}$  be the weight raising/lowering operator given by

$$V_{\pm} := \frac{1}{(-8\pi)} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \sqrt{-1} \right) \in \text{Lie}(\text{GL}_2(\mathbf{R})) \otimes_{\mathbf{R}} \mathbf{C}.$$

For each integer  $k$  we denote by  $\sigma_k$  the (limit of) discrete series of  $\text{GL}_2(\mathbf{R})$  of the minimal weight  $\pm k$  and by  $W_k$  the Whittaker function of  $\sigma_k$  characterized by

$$W_k(\text{diag}(y, 1)) = y^{k/2} e^{-2\pi y} \mathbb{I}_{\mathbf{R}_+}(y).$$

Set  $W_k^{[t]} = V_{\pm}^t W_k$ . It follows from (6.2) below that

$$(4.5) \quad W_k^{[t]}(\text{diag}(y, 1)) = \sum_{j=0}^t (-4\pi)^{j-t} \binom{t}{j} \frac{\Gamma(t+k)}{\Gamma(j+k)} \cdot y^{\frac{k}{2}+j} e^{-2\pi y} \mathbb{I}_{\mathbf{R}_+}(y).$$

Fix a triplet  $(k, l, m)$  of positive integers such that  $k \geq l \geq m$  and  $k < l + m$ . Put  $\mathcal{J}_{\infty} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . Define

$$Z_{\infty}(s) := Z \left( \rho(\mathcal{J}_{\infty}) W_k, \rho(\mathcal{J}_{\infty}) W_l^{\left[\frac{k-l-\lambda_2}{2}\right]}, \rho(\mathcal{J}_{\infty}) W_m^{\left[\frac{k-m-\lambda_3}{2}\right]}, f_{s, \infty}^{[k, \lambda]} \right),$$

where  $\lambda$  is the parity type of  $(k, l, m)$ . Recall that

$$\begin{aligned} L(s, \sigma_k \times \sigma_l \times \sigma_m) &= \Gamma_{\mathbf{C}} \left( s + \frac{k+l+m-3}{2} \right) \Gamma_{\mathbf{C}} \left( s + \frac{k-l+m-1}{2} \right) \\ &\quad \times \Gamma_{\mathbf{C}} \left( s + \frac{k+l-m-1}{2} \right) \Gamma_{\mathbf{C}} \left( s + \frac{m+l-k-1}{2} \right). \end{aligned}$$

Put

$$\gamma_{(k, m, l)}^*(s) = (\sqrt{-1})^{k+2\lambda_2+\lambda_1} \frac{\Gamma(s + \frac{k-m-l}{2} + 1)}{\Gamma(s - \frac{k-\lambda_1}{2} + \lambda_2 + 1)} \cdot \frac{\Gamma(s + \frac{k+\lambda_1}{2})}{\Gamma(s + \frac{k+\lambda_1}{2} + 1)} \cdot \frac{\pi^{3s+1} (4\pi)^{l+m - \frac{k-\lambda_1}{2} + \lambda_2}}{4\Gamma(s + \frac{m+l-k}{2}) \Gamma(2s+k)}.$$

**Lemma 4.8.** *If  $\lambda$  is the parity type of  $(k, l, m)$ , then*

$$Z_{\infty}(s) = (\chi_{\infty} \hat{\omega}_{\infty})(-1) \text{vol}(\text{SO}(2))^3 \cdot \frac{\gamma_{(k, m, l)}^*(s)}{2^{5+(k+m+l)}} L \left( s + \frac{1}{2}, \sigma_k \times \sigma_l \times \sigma_m \right).$$

PROOF. For  $a = (a_1, a_2, a_3) \in \mathbf{R}_+^3$  and  $x \in \mathbf{R}$ , we set

$$\begin{aligned} z &= x + \sqrt{-1}(a_1^2 + a_2^2 + a_3^2), \\ t(a) &= \text{diag} \left( \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix}, \begin{pmatrix} a_3 & 0 \\ 0 & a_3^{-1} \end{pmatrix} \right), \\ u(x) &= \text{diag} \left( \begin{pmatrix} 1 & \frac{x}{3} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{x}{3} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{x}{3} \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

When  $x \neq 0$ , the Iwasawa decomposition of  $\eta u(x)t(a)$  can be described as follows: Put

$$P = \begin{pmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2^2 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^2 \end{pmatrix}.$$

We write  $\eta u(x)t(a) = \mathbf{n}(z) \mathbf{m}(A) \mathbf{u}$  with  $z \in \text{Sym}_3(\mathbf{R})$ ,  $A \in \text{GL}_3(\mathbf{R})$  and  $\mathbf{u} = \begin{pmatrix} D & -C \\ C & D \end{pmatrix} \in \text{U}(3)$ . Since  $D^{-1}C = x^{-1}P$ , we can choose  $U \in \text{GL}_3(\mathbf{R})$  so that

$$U^t U = (x^2 \mathbf{1}_3 + P^2)^{-1}, \quad \mathbf{u} = \begin{pmatrix} Ux & -UP \\ UP & Ux \end{pmatrix} \in \text{U}(3).$$

Put  $u = Ux - \sqrt{-1}UP$ . Then  $u^t u = (x \mathbf{1}_3 - P\sqrt{-1})(x \mathbf{1}_3 + P\sqrt{-1})^{-1}$ . By direct computations we get

$$\det A = a_1 a_2 a_3 |z|^{-1}, \quad \det u = \frac{\bar{z}}{|z|}, \quad H_{23}(u) = -2\sqrt{-1} \frac{a_2 a_3}{z}, \quad H_{12}(u) = -2\sqrt{-1} \frac{a_1 a_2}{z}$$

(see [GK92, (6.7), (6.8)]). Put

$$b = \frac{1}{2}(k - l - \lambda_2), \quad c = \frac{1}{2}(k - m - \lambda_3), \quad (\mathbf{s}, \mathbf{k}, \mathbf{l}, \mathbf{m}) = \left( s + \frac{\lambda_3}{2}, k - \lambda_2, l, m - \lambda_1 \right).$$

It follows that

$$\begin{aligned} f_{s,\infty}^{[k,\lambda]}(\boldsymbol{\eta}t(u(x)t(a)\mathbf{d}(-1))) &= \chi_\infty(-1)(-1)^{k-\lambda_1} \left( 2\sqrt{-1} \frac{a_1 a_2}{\bar{z}} \right)^{\lambda_1} \left( -2\sqrt{-1} \frac{a_2 a_3}{z} \right)^{\lambda_2} \left( \frac{a_1 a_2 a_3}{|z|} \right)^{2s+2} \left( \frac{z}{|z|} \right)^{k-\lambda_1} \\ &= \chi_\infty(-1) 2^{\lambda_1+\lambda_2} \sqrt{-1}^{\lambda_2-\lambda_1} (a_1 a_2 a_3)^{2s+2} a_1^{-\lambda_2} a_3^{-\lambda_1} |z|^{-2s-2-\mathbf{k}} (-z)^{\mathbf{k}}. \end{aligned}$$

From (4.2) and (4.5)  $\frac{2Z_\infty(s)}{\text{vol}(\text{SO}(2))^3}$  equals

$$\begin{aligned} & \int_{\mathbf{R}} \int_{\mathbf{R}_+^3} W_k(\mathbf{n}(x/3)\mathbf{m}(a_1)) W_l^{[b]}(\mathbf{n}(x/3)\mathbf{m}(a_2)) W_m^{[c]}(\mathbf{n}(x/3)\mathbf{m}(a_3)) f_{s,\infty}^{[k,\lambda]}(\boldsymbol{\eta}t(u(x)t(a)\mathbf{d}(-1))) dx \prod_{j=1}^3 \frac{d^\times a_j}{|a_j|^2} \\ &= \chi_\infty(-1) 2^{\lambda_1+\lambda_2} \sqrt{-1}^{\lambda_2-\lambda_1} (-4\pi)^{-b-c} \sum_{A=0}^{\infty} \sum_{B=0}^{\infty} (-4\pi)^{A+B} \binom{b}{A} \binom{c}{B} \frac{\Gamma(l+b)}{\Gamma(l+A)} \frac{\Gamma(m+c)}{\Gamma(m+B)} \\ & \quad \times \int_{\mathbf{R}} \int_{\mathbf{R}_+^3} a_1^{2s+\mathbf{k}} a_2^{2s+1+2A} a_3^{2s+\mathbf{m}+2B} |z|^{-2s-2-\mathbf{k}} (-z)^{\mathbf{k}} e^{2\pi\sqrt{-1}x} e^{-2\pi(a_1^2+a_2^2+a_3^2)} dx \prod_{j=1}^3 d^\times a_j. \end{aligned}$$

Put  $\alpha = \mathbf{s} + 1 + \frac{\mathbf{k}}{2}$  and  $\beta = \mathbf{s} + 1 - \frac{\mathbf{k}}{2}$ . The last integral equals

$$\frac{(-2\pi\sqrt{-1})^\alpha (2\pi\sqrt{-1})^\beta}{\Gamma(\alpha)\Gamma(\beta)} \int_{\mathbf{R}_+^4} a_1^{\mathbf{k}+2s} a_2^{1+2A+2s} a_3^{\mathbf{m}+2B+2s} \frac{(1+t)^{\alpha-1} t^{\beta-1}}{e^{4\pi(a_1^2+a_2^2+a_3^2)}(1+t)} dt \prod_{j=1}^3 d^\times a_j.$$

We here use the identity

$$\int_{\mathbf{R}} \frac{e^{-2\pi\sqrt{-1}x} dx}{(x + \sqrt{-1}y)^\alpha (x - \sqrt{-1}y)^\beta} = \frac{(-2\pi\sqrt{-1})^\alpha (2\pi\sqrt{-1})^\beta}{\Gamma(\alpha)\Gamma(\beta)} \int_{\mathbf{R}_+} \frac{(t+1)^{\alpha-1} t^{\beta-1}}{e^{2\pi y(1+2t)}} dt$$

(see [GK92, (6.11)]). The quadruple integral above equals

$$\begin{aligned} & \frac{1}{(4\pi)^{\frac{\mathbf{k}+1+\mathbf{m}}{2}+3s+A+B}} \int_{\mathbf{R}_+^4} \frac{a_1^{\mathbf{k}+2s} a_2^{1+2A+2s} a_3^{\mathbf{m}+2B+2s} (1+t)^{\alpha-1} t^{\beta-1}}{e^{a_1^2+a_2^2+a_3^2} (1+t)^{\frac{\mathbf{k}+1+\mathbf{m}}{2}+3s+A+B}} dt \prod_{j=1}^3 d^\times a_j \\ &= \frac{\Gamma(\frac{\mathbf{k}}{2} + \mathbf{s}) \Gamma(\frac{1}{2} + \mathbf{s} + A) \Gamma(\frac{\mathbf{m}}{2} + \mathbf{s} + B)}{2^3 (4\pi)^{\frac{\mathbf{k}+1+\mathbf{m}}{2}+3s+A+B}} \int_{\mathbf{R}_+} \frac{(1+t)^{\alpha-1} t^{\beta-1}}{(1+t)^{\frac{\mathbf{k}+1+\mathbf{m}}{2}+3s+A+B}} dt. \end{aligned}$$

Recall that

$$\begin{aligned} \int_0^\infty \frac{(1+t)^{\alpha-1} t^{\beta-1}}{(1+t)^{\frac{\mathbf{k}+1+\mathbf{m}}{2}+3s+A+B}} dt &= B \left( \beta, 1 - \alpha - \beta + \frac{\mathbf{k}+1+\mathbf{m}}{2} + 3s + A + B \right) \\ &= \frac{\Gamma(\beta)\Gamma(1 - \alpha - \beta + \frac{\mathbf{k}+1+\mathbf{m}}{2} + 3s + A + B)}{\Gamma(1 - \alpha + \frac{\mathbf{k}+1+\mathbf{m}}{2} + 3s + A + B)}. \end{aligned}$$

We finally get

$$\begin{aligned} Z_\infty(s) &= \text{vol}(\text{SO}(2))^3 \chi_\infty(-1) 2^{\lambda_1+2\lambda_2-2-4s-3k} \pi^{2-s+\frac{\lambda_2+\lambda_3-3k}{2}} (-\sqrt{-1})^{k+\lambda_1} (-1)^{\lambda_2+b+c} \\ & \quad \times \frac{\Gamma(\mathbf{s} + \frac{\mathbf{k}}{2})}{\Gamma(\mathbf{s} + \frac{\mathbf{k}}{2} + 1)} \Gamma(l+b)\Gamma(m+c) \sum_{A,B} (-1)^{A+B} \binom{b}{A} \binom{c}{B} \frac{\Gamma_\infty(s; A, B)}{\Gamma(l+A)\Gamma(m+B)}, \end{aligned}$$

where

$$\Gamma_\infty(\mathbf{s}; A, B) = \frac{\Gamma(\mathbf{s} + \frac{1}{2} + A) \Gamma(\mathbf{s} + \frac{\mathbf{m}}{2} + B) \Gamma(\mathbf{s} + \frac{\mathbf{k}+1+\mathbf{m}}{2} - 1 + A + B)}{\Gamma(2\mathbf{s} + \frac{1+\mathbf{m}}{2} + A + B)}.$$



Lemma 3 of [Orl87] with  $\alpha = l = 1$ ,  $t = \mathbf{s} + \frac{l}{2}$ ,  $\beta = B + \frac{\mathbf{m}-l}{2}$  and  $N = b$  gives

$$\begin{aligned} & \Gamma(l+b) \sum_{A=0}^b (-1)^A \binom{b}{A} \frac{\Gamma(\mathbf{s} + \frac{l}{2} + A) \Gamma(\mathbf{s} + \frac{\mathbf{k}+l+\mathbf{m}}{2} - 1 + A + B)}{\Gamma(l+A) \Gamma(2\mathbf{s} + \frac{l+\mathbf{m}}{2} + A + B)} \\ &= (-1)^b \frac{\Gamma(\mathbf{s} + \frac{l}{2}) \Gamma(\mathbf{s} + B + \frac{\mathbf{m}}{2} + l + b - 1) \Gamma(\mathbf{s} + B + \frac{\mathbf{m}}{2} + b) \Gamma(\mathbf{s} - \frac{l}{2} + 1)}{\Gamma(2\mathbf{s} + B + \frac{\mathbf{m}+l}{2} + b) \Gamma(\mathbf{s} + B + \frac{\mathbf{m}}{2}) \Gamma(\mathbf{s} - \frac{l}{2} - b + 1)}. \end{aligned}$$

It follows that

$$\begin{aligned} & \Gamma(l+b) \Gamma(m+c) \sum_{A,B} (-1)^{A+B} \binom{b}{A} \binom{c}{B} \frac{\Gamma_\infty(s; A, B)}{\Gamma(l+A) \Gamma(m+B)} \\ &= (-1)^b \frac{\Gamma(\mathbf{s} + \frac{l}{2}) \Gamma(\mathbf{s} - \frac{l}{2} + 1)}{\Gamma(\mathbf{s} - \frac{l}{2} - b + 1)} \Gamma(m+c) \sum_B (-1)^B \binom{c}{B} \frac{\Gamma(\mathbf{s} + B + \frac{\mathbf{m}}{2} + l + b - 1) \Gamma(\mathbf{s} + B + \frac{\mathbf{m}}{2} + b)}{\Gamma(2\mathbf{s} + B + \frac{\mathbf{m}+l}{2} + b) \Gamma(m+B)}. \end{aligned}$$

Again we apply Lemma 3 of [Orl87] with  $\alpha = m$ ,  $t = \mathbf{s} + \frac{\mathbf{m}}{2} + b$ ,  $\beta = \frac{l-\mathbf{m}}{2} - b$  and  $N = c$  to obtain

$$\begin{aligned} & \Gamma(m+c) \sum_B (-1)^B \binom{c}{B} \frac{\Gamma(\mathbf{s} + B + \frac{\mathbf{m}}{2} + l + b - 1) \Gamma(\mathbf{s} + B + \frac{\mathbf{m}}{2} + b)}{\Gamma(2\mathbf{s} + B + \frac{\mathbf{m}+l}{2} + b) \Gamma(m+B)} \\ &= (-1)^c \frac{\Gamma(\mathbf{s} + \frac{\mathbf{m}}{2} + b) \Gamma(\mathbf{s} + \frac{l}{2} + m + c - 1) \Gamma(\mathbf{s} + \frac{l}{2} + c) \Gamma(\mathbf{s} + \frac{\mathbf{m}}{2} + b - m + 1)}{\Gamma(2\mathbf{s} + \frac{\mathbf{m}+l}{2} + b + c) \Gamma(\mathbf{s} + \frac{l}{2}) \Gamma(\mathbf{s} + \frac{\mathbf{m}}{2} - m + b - c + 1)}. \end{aligned}$$

Then we can see that the double summation equals

$$\begin{aligned} & (-1)^{b+c} \frac{\Gamma(\mathbf{s} + \frac{\mathbf{m}}{2} + b) \Gamma(\mathbf{s} + \frac{l}{2} + m + c - 1) \Gamma(\mathbf{s} + \frac{l}{2} + c) \Gamma(\mathbf{s} + c - \frac{l}{2} + 1)}{\Gamma(\mathbf{s} - \frac{l}{2} - b + 1) \Gamma(2\mathbf{s} + \frac{\mathbf{m}+l}{2} + b + c)} \\ &= (-1)^{b+c} \frac{\Gamma(\mathbf{s} + \frac{\mathbf{k}-l+\mathbf{m}}{2}) \Gamma(\mathbf{s} + \frac{\mathbf{k}+l+\mathbf{m}}{2} - 1) \Gamma(\mathbf{s} + \frac{\mathbf{k}-l-\mathbf{m}}{2} + 1) \Gamma(\mathbf{s} + \frac{\mathbf{k}-m+l}{2})}{\Gamma(2\mathbf{s} + \mathbf{k})} \cdot \frac{1}{\Gamma(\mathbf{s} - \frac{\mathbf{k}}{2} + 1)}. \end{aligned}$$

The last equality uses  $b = \frac{\mathbf{k}-1}{2}$ ,  $m+c = \frac{\mathbf{k}+\mathbf{m}}{2}$ ,  $\mathbf{s} + c = \mathbf{s} + \frac{\mathbf{k}-m}{2}$  and  $2\mathbf{s} + \mathbf{k} + \mathbf{m} = 2\mathbf{s} + \mathbf{k} + m$ .  $\square$

## 5. CLASSICAL AND $p$ -ADIC MODULAR FORMS

**5.1. Conventions.** Besides the standard symbols  $\mathbf{Z}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$ ,  $\mathbf{C}$ ,  $\mathbf{Z}_\ell$ ,  $\mathbf{Q}_\ell$  we denote by  $\mathbf{R}_+$  the group of strictly positive real numbers. Fix algebraic closures of  $\mathbf{Q}$  and  $\mathbf{Q}_p$ , denoting them by  $\overline{\mathbf{Q}}$  and  $\overline{\mathbf{Q}}_p$ . Let  $\mathbf{A}$  be the ring of adèles of  $\mathbf{Q}$  and  $\mu_n$  the group of  $n$ -th roots of unity in  $\overline{\mathbf{Q}}$ . Given a place  $v$  of  $\mathbf{Q}$ , we write  $\mathbf{Q}_v$  for the completion of  $\mathbf{Q}$  with respect to  $v$ . We shall regard  $\mathbf{Q}_v$  and  $\mathbf{Q}_v^\times$  as subgroups of  $\mathbf{A}$  and  $\mathbf{A}^\times$  in a natural way. We denote by the formal symbol  $\infty$  the real place of  $\mathbf{Q}$  and do not use  $\ell$  for the infinite place. Let  $\psi_{\mathbf{Q}} : \mathbf{A}/\mathbf{Q} \rightarrow \mathbf{C}^\times$  be the additive character with the archimedean component  $\psi_\infty(x) = e^{2\pi\sqrt{-1}x}$  and  $\psi_\ell : \mathbf{Q}_\ell \rightarrow \mathbf{C}^\times$  the local component of  $\psi_{\mathbf{Q}}$  at  $\ell$ .

Denote by  $\alpha_{\mathbf{Q}_v} = |\cdot|_v$  the absolute value on  $\mathbf{Q}_v$  normalized so that  $\alpha_{\mathbf{R}}$  is the usual absolute value on  $\mathbf{R}$ , and  $|\ell|_\ell = \ell^{-1}$  if  $v = \ell$  is finite. For  $a \in \mathbf{A}^\times$ , let  $a_v \in \mathbf{Q}_v^\times$  denote the  $v$ -component of  $a$ . Define the character  $\alpha_{\mathbf{A}} = \mathbf{Q}^\times \backslash \mathbf{A}^\times \rightarrow \mathbf{R}_+$  by  $\alpha_{\mathbf{A}}(a) = |a|_{\mathbf{A}} = \prod_v |a_v|_v$ . Recall the local Riemann zeta functions

$$\zeta_\infty(s) = \pi^{-s/2} \Gamma(s/2), \quad \zeta_\ell(s) = (1 - \ell^{-s})^{-1}.$$

Define the completed Riemann zeta function  $\zeta_{\mathbf{Q}}(s)$  by  $\zeta_{\mathbf{Q}}(s) = \prod_v \zeta_v(s)$ . In particular,  $\zeta_{\mathbf{Q}}(2) = \frac{\pi}{6}$ . For each rational prime  $\ell$ , let  $v_\ell : \mathbf{Q}_\ell^\times \rightarrow \mathbf{Z}$  denote the valuation normalized so that  $v_\ell(\ell) = 1$ . To avoid possible confusion, denote by  $\varpi_\ell = (\varpi_{\ell,v}) \in \mathbf{A}^\times$  the idèle defined by  $\varpi_{\ell,\ell} = \ell$  and  $\varpi_{\ell,v} = 1$  if  $v \neq \ell$ .

If  $\omega : \mathbf{A}^\times \rightarrow \mathbf{C}^\times$  is a quasi-character, then we denote by  $\omega_v : \mathbf{Q}_v^\times \rightarrow \mathbf{C}^\times$  the local component of  $\omega$  at  $v$ . If  $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \rightarrow \overline{\mathbf{Q}}^\times$  is a Dirichlet character modulo  $N$ , then we denote the  $\ell$ -exponent of the conductor of  $\chi$  by  $c_\ell(\chi) \leq v_\ell(N)$ . We can associate to a Dirichlet character  $\chi$  of conductor  $N$  a Hecke character  $\chi_{\mathbf{A}}$ , called the *adèlic lift* of  $\chi$ , which is the unique finite order Hecke character  $\chi_{\mathbf{A}} : \mathbf{Q}^\times \mathbf{R}_+ \backslash \mathbf{A}^\times / (1 + N\widehat{\mathbf{Z}}) \cap \widehat{\mathbf{Z}}^\times \rightarrow \overline{\mathbf{Q}}^\times$  of conductor  $N$  such that  $\chi_{\mathbf{A}}(\varpi_\ell) = \chi(\ell)^{-1}$  for any prime number  $\ell \nmid N$ .

Fix an odd prime number  $p$  and an isomorphism  $\iota_p : \overline{\mathbf{Q}}_p \simeq \mathbf{C}$  once and for all.

**Definition 5.1** (Teichmüller and cyclotomic characters). The action of  $G_{\mathbf{Q}}$  on  $\mu_{p^\infty} := \varinjlim \mu_{p^n}$  gives rise to a continuous homomorphism  $\varepsilon_{\text{cyc}} : G_{\mathbf{Q}} \rightarrow \mathbf{Z}_p^\times$ , called the  $p$ -adic cyclotomic character, defined by  $\sigma(\zeta) = \zeta^{\varepsilon_{\text{cyc}}(\sigma)}$  for every  $\zeta \in \mu_{p^\infty}$ . The character  $\varepsilon_{\text{cyc}}$  splits into the  $p$ -adic Teichmüller character  $\omega : G_{\mathbf{Q}} \rightarrow \text{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q}) \rightarrow \mathbf{Z}_p^\times$  and  $\langle \cdot \rangle : G_{\mathbf{Q}} \rightarrow \text{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \xrightarrow{\sim} 1 + p\mathbf{Z}_p$ . The character  $\omega$  sends  $\sigma$  to the unique solution in  $\mathbf{Z}_p^\times$  of  $\omega(\sigma)^p = \omega(\sigma) \equiv \varepsilon_{\text{cyc}}(\sigma) \pmod{p}$ . We often regard  $\omega$  and  $\langle \cdot \rangle^s$  with  $s \in \mathbf{Z}_p$  as characters of  $\mathbf{Z}_p^\times$ . We sometimes identify  $\omega$  with the Dirichlet character  $\iota_p \circ \omega : (\mathbf{Z}/p\mathbf{Z})^\times \rightarrow \mathbf{C}^\times$ .

**Remark 5.2.** (1) Let  $\chi$  be a Dirichlet character. If  $\chi_v$  stands for the restriction of  $\chi_{\mathbf{A}}$  to  $\mathbf{Q}_v^\times$ , then  $\chi_\ell(\ell) = \chi(\ell)^{-1}$  for each prime number  $\ell \nmid N$ . Furthermore, if  $N$  is a power of  $p$  and  $b$  is not divisible by  $p$ , then  $\chi_p(b) = \chi(b)$ .

(2) Let  $\chi$  be a character of  $\mathbf{Z}_p^\times$  of finite order, which can be regarded as either a complex character or a  $p$ -adic character via composition with  $\iota_p$ . We view  $\chi$  as a character of  $G_{\mathbf{Q}}$  via composition with the cyclotomic character  $\varepsilon_{\text{cyc}}$ . Let  $\mathbf{Q}^{\text{ab}} = \bigcup_{N=1}^\infty \mathbf{Q}(\mu_N)$  be the maximal abelian extension of  $\mathbf{Q}$  and

$$\text{rec}_{\mathbf{Q}} : \mathbf{Q}^\times \mathbf{R}_+ \backslash \mathbf{A}^\times \xrightarrow{\sim} \text{Gal}(\mathbf{Q}^{\text{ab}}/\mathbf{Q})$$

the geometrically normalized reciprocity law map, i.e.,  $\text{rec}_{\mathbf{Q}}(\varpi_\ell)|_{\mathbf{Q}(\mu_{p^\infty})} = \text{Frob}_\ell$  for  $\ell \neq p$ . Since  $\chi$  factors through the quotient  $\mathbf{Z}_p^\times \twoheadrightarrow (\mathbf{Z}/p^{c(\chi)}\mathbf{Z})^\times$ , we can identify  $\chi$  with a Dirichlet character of  $p$ -power conductor. Then since  $\chi_{\mathbf{A}}(\varpi_\ell) = \chi(\ell)^{-1} = \chi(\varepsilon_{\text{cyc}}(\text{Frob}_\ell))$  for  $\ell \neq p$ ,

$$\chi_{\mathbf{A}} = \chi \circ \varepsilon_{\text{cyc}} \circ \text{rec}_{\mathbf{Q}}, \quad \chi_p|_{\mathbf{Z}_p^\times} = \chi.$$

**5.2. Differential operators and nearly holomorphic modular forms.** Let  $\text{GL}_2^+(\mathbf{R})$  be the subgroup of  $\text{GL}_2(\mathbf{R})$  consisting of matrices with positive determinant and  $\mathfrak{H}_1$  the upper half plane on which  $\text{GL}_2^+(\mathbf{R})$  acts via fractional transformation. Define a subgroup of  $\text{SL}_2(\mathbf{Z})$  of finite index

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbf{Z}) \mid N|c \right\}.$$

The Lie group  $\text{GL}_2^+(\mathbf{R})$  acts on the complex vector space of complex valued functions  $f$  on  $\mathfrak{H}_1$  as in (4.1).

The Maass-Shimura differential operators  $\delta_k$  and  $\lambda_z$  on  $C^\infty(\mathfrak{H}_1)$  are given by

$$\delta_k = \frac{1}{2\pi\sqrt{-1}} \left( \frac{\partial}{\partial z} + \frac{k}{2\sqrt{-1}y} \right), \quad \lambda_z = -\frac{1}{2\pi\sqrt{-1}} y^2 \frac{\partial}{\partial \bar{z}}$$

with  $y = \text{Im } z \in \mathbf{R}_+$ . Let  $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \rightarrow \mathbf{C}^\times$  be a Dirichlet character, which we extend to a character  $\chi^\downarrow : \Gamma_0(N) \rightarrow \mathbf{C}^\times$  by  $\chi^\downarrow \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \chi(d)$ . For a non-negative integer  $m$  the space  $\mathcal{N}_k^{[m]}(N, \chi)$  of nearly holomorphic modular forms of weight  $k$ , level  $N$  and character  $\chi$  consists of slowly increasing functions  $f \in C^\infty(\mathfrak{H}_1)$  such that  $\lambda_z^{m+1} f = 0$  and  $f|_k \gamma = \chi^\downarrow(\gamma) f$  for  $\gamma \in \Gamma_0(N)$  (cf. [Hid93, page 314]). Put  $\mathcal{N}_k(N, \chi) = \bigcup_{m=0}^\infty \mathcal{N}_k^{[m]}(N, \chi)$  (cf. [Hid93, (1a), page 310]). By definition  $\mathcal{N}_k^{[0]}(N, \chi) = \mathcal{M}_k(N, \chi)$  is the space of elliptic modular forms of weight  $k$ , level  $N$  and character  $\chi$ . Denote the space of elliptic cusp forms in  $\mathcal{M}_k(N, \chi)$  by  $\mathcal{S}_k(N, \chi)$ . Put  $\delta_k^m = \delta_{k+2m-2} \cdots \delta_{k+2}\delta_k$ . If  $f \in \mathcal{N}_k(N, \chi)$ , then  $\delta_k^m f \in \mathcal{N}_{k+2m}(N, \chi)$  (see [Hid93, page 312]).

Define an open compact subgroup of  $\text{GL}_2(\widehat{\mathbf{Z}})$  by

$$U_0(N) = \left\{ g \in \text{GL}_2(\widehat{\mathbf{Z}}) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N\widehat{\mathbf{Z}}} \right\}.$$

We extend  $\chi_{\mathbf{A}}$  to a character  $\chi_{\mathbf{A}}^\downarrow$  of  $U_0(N)$  by  $\chi_{\mathbf{A}}^\downarrow(g) = \prod_{\ell|N} \chi_\ell^\downarrow(g_\ell)$  (see (2.4) for the definition of  $\chi_\ell^\downarrow$ ). Let  $\mathcal{A}_k(N, \chi_{\mathbf{A}}^{-1})$  be the space of functions  $\Phi : \text{GL}_2(\mathbf{A}) \rightarrow \mathbf{C}$  such that  $V_-^m \Phi = 0$  for some  $m$  and such that

$$\Phi(z\gamma g \kappa_\theta u) = \chi_{\mathbf{A}}(z)^{-1} \Phi(g) e^{\sqrt{-1}k\theta} \chi_{\mathbf{A}}^\downarrow(u)^{-1} \quad (z \in \mathbf{A}^\times, \gamma \in \text{GL}_2(\mathbf{Q}), \theta \in \mathbf{R}, u \in U_0(N)).$$

**Definition 5.3** (The adèlic lift). With each nearly holomorphic modular form  $f \in \mathcal{N}_k(N, \chi)$  we can associate a unique automorphic form  $\Phi(f) \in \mathcal{A}_k(N, \chi_{\mathbf{A}}^{-1})$  defined by the equation

$$\Phi(f)(\gamma g_\infty u) := (f|_k g_\infty)(\sqrt{-1}) \cdot \chi_{\mathbf{A}}^\downarrow(u)^{-1}$$

for  $\gamma \in \mathrm{GL}_2(\mathbf{Q})$ ,  $g_\infty \in \mathrm{GL}_2^+(\mathbf{R})$  and  $u \in U_0(N)$  (cf. [Cas73, §3]). We call  $\Phi(f)$  the *adèlic lift* of  $f$ . Conversely, we can recover  $f$  from  $\Phi(f)$  by

$$f(x + \sqrt{-1}y) = y^{-k/2} \Phi(f) \left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right).$$

Recall that  $V_\pm$  are the operators as defined in §4.4. By definition we have

$$\Phi(\delta_k f) = V_+ \Phi(f), \quad \Phi(\lambda_z f) = V_- \Phi(f).$$

We define the Whittaker coefficient and the constant term of  $\Phi \in \mathcal{A}_k(N, \chi_{\mathbf{A}}^{-1})$  by

$$W(g, \Phi) = \int_{\mathbf{Q} \backslash \mathbf{A}} \Phi(\mathbf{n}(x)g) \overline{\psi_{\mathbf{Q}}(x)} dx, \quad \mathbf{a}_0(g, \Phi) = \int_{\mathbf{Q} \backslash \mathbf{A}} \Phi(\mathbf{n}(x)g) dx.$$

**5.3. Ordinary  $\mathbf{I}$ -adic modular forms.** For any subring  $A \subset \mathbf{C}$  the space  $\mathcal{S}_k(N, \chi; A)$  consists of elliptic cusp forms  $f = \sum_{n=1}^{\infty} \mathbf{a}(n, f)q^n \in \mathcal{S}_k(N, \chi)$  such that  $\mathbf{a}(n, f) \in A$  for all  $n$ . For every subring  $A \subset \overline{\mathbf{Q}}_p$  containing  $\mathbf{Z}[\chi]$  we define the space of cusp forms over  $A$  by

$$\mathcal{S}_k(N, \chi; A) = \mathcal{S}_k(N, \chi; \mathbf{Z}[\chi]) \otimes_{\mathbf{Z}[\chi]} A.$$

Here we view  $\chi$  as a  $p$ -adic Dirichlet character via  $\iota_p^{-1}$ .

**Definition 5.4** ( *$p$ -stabilized newforms*). We say that a normalized Hecke eigenform  $f \in \mathcal{S}_k(Np, \chi)$  is an (ordinary)  $p$ -stabilized newform (with respect to  $\iota_p : \mathbf{C} \simeq \overline{\mathbf{Q}}_p$ ) if  $f$  is new outside  $p$  and the eigenvalue of  $U_p$ , i.e. the  $p$ -th Fourier coefficient  $\iota_p(\mathbf{a}(p, f))$ , is a  $p$ -adic unit. The prime-to- $p$  part  $N'$  of the conductor of  $f$  is called the tame conductor of  $f$ . There is a unique decomposition  $\chi = \chi' \omega^a \epsilon$  with  $a \in \mathbf{Z}/(p-1)\mathbf{Z}$ , where  $\chi'$  is a Dirichlet character modulo  $N'$  and  $\epsilon$  is a character of  $1 + p\mathbf{Z}_p$ . We call  $\chi' \omega^a$  the tame nebentypus of  $f$ .

Let  $f^\circ = \sum_{n=1}^{\infty} \mathbf{a}(n, f^\circ)q^n \in \mathcal{S}_k(Np, \chi)$  be a primitive Hecke eigenform of conductor  $N_{f^\circ}$ . We call  $f^\circ$  ordinary if  $\iota_p^{-1}(\mathbf{a}(p, f^\circ))$  is a  $p$ -adic unit. If this is the case, then precisely one of the roots of the polynomial  $X^2 - \mathbf{a}(p, f^\circ)X + \chi(p)p^{k-1}$  (call it  $\alpha_p(f)$ ) satisfies  $|\iota_p(\alpha_p(f))|_p = 1$ . We associate to an ordinary primitive form  $f^\circ$  the  $p$ -stabilized newform by

$$(5.1) \quad f(\tau) = f^\circ(\tau) - \frac{\chi(p)p^{k-1}}{\alpha_p(f)} f^\circ(p\tau) \in \mathcal{S}_k(N_{f^\circ}p, \chi),$$

if  $N_{f^\circ}$  and  $p$  are coprime, and  $f = f^\circ$  if  $p$  divides  $N_{f^\circ}$ .

Let  $\mathcal{O}$  be the ring of integers of a finite extension of  $\mathbf{Q}_p$  and  $\mathbf{I}$  a normal domain finite flat over  $\Lambda = \mathcal{O}[[1 + p\mathbf{Z}_p]]$ . A point  $Q \in \mathrm{Spec} \mathbf{I}(\overline{\mathbf{Q}}_p)$ , a ring homomorphism  $Q : \mathbf{I} \rightarrow \overline{\mathbf{Q}}_p$ , is said to be locally algebraic if the restriction of  $Q$  to  $1 + p\mathbf{Z}_p$  is of the form  $Q(z) = z^{k_Q} \epsilon_Q(z)$  with  $k_Q$  an integer and  $\epsilon_Q(z) \in \mu_{p^\infty}$ . We shall call  $k_Q$  the *weight* of  $Q$  and  $\epsilon_Q$  the *finite part* of  $Q$ . Let  $\mathfrak{X}_{\mathbf{I}}$  be the set of locally algebraic points  $Q \in \mathrm{Spec} \mathbf{I}(\overline{\mathbf{Q}}_p)$  of weight  $k_Q \geq 1$ . A point  $Q \in \mathfrak{X}_{\mathbf{I}}$  is said to be *arithmetic* if  $k_Q \geq 2$ . Let  $\mathfrak{X}_{\mathbf{I}}^+$  be the set of arithmetic points,  $\wp_Q = \mathrm{Ker} Q$  the prime ideal of  $\mathbf{I}$  corresponding to  $Q$  and  $\mathcal{O}(Q)$  the image of  $\mathbf{I}$  under  $Q$ .

Let  $N$  be a positive integer prime to  $p$  and  $\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \rightarrow \mathcal{O}^\times$  a Dirichlet character modulo  $Np$ . An  $\mathbf{I}$ -adic cusp form is a formal power series  $\mathbf{f}(q) = \sum_{n=1}^{\infty} \mathbf{a}(n, \mathbf{f})q^n \in \mathbf{I}[[q]]$  with the following property: there exists an integer  $a_{\mathbf{f}}$  such that for arithmetic points  $Q \in \mathfrak{X}_{\mathbf{I}}^+$  with  $k_Q \geq a_{\mathbf{f}}$ , the specialization  $\mathbf{f}_Q(q) = \sum_{n=1}^{\infty} Q(\mathbf{a}(n, \mathbf{f}))q^n$  is the Fourier expansion of a cusp form  $\mathbf{f}_Q \in \mathcal{S}_{k_Q}(Np^e, \chi \omega^{-k_Q} \epsilon_Q; \mathcal{O}(Q))$ . Denote by  $\mathbf{S}(N, \chi, \mathbf{I})$  the space of  $\mathbf{I}$ -adic cusp forms of tame level  $N$  and (even) branch character  $\chi$ . This space  $\mathbf{S}(N, \chi, \mathbf{I})$  is equipped with the action of the Hecke operators  $T_\ell$  for  $\ell \nmid Np$  as in [Wil88, page 537] and the operators  $\mathbf{U}_\ell$  for  $\ell \mid pN$  given by  $\mathbf{U}_\ell(\sum_n \mathbf{a}(n, \mathbf{f})q^n) = \sum_n \mathbf{a}(n\ell, \mathbf{f})q^n$ .

Hida's ordinary projector  $e_{\mathrm{ord}}$  is defined by

$$e_{\mathrm{ord}} := \lim_{n \rightarrow \infty} \mathbf{U}_p^{n!}.$$

It has a well-defined action on the space of classical modular forms preserving the cuspidal part as well as on the space  $\mathbf{S}(N, \chi, \mathbf{I})$  (cf. [Wil88, page 537 and Proposition 1.2.1]). The space  $\mathbf{S}^{\mathrm{ord}}(N, \chi, \mathbf{I}) := e_{\mathrm{ord}} \mathbf{S}(N, \chi, \mathbf{I})$  is called the space of ordinary  $\mathbf{I}$ -adic forms with respect to  $\chi$ . Put

$$\mathcal{M}_k^{\mathrm{ord}}(N, \chi; A) = e_{\mathrm{ord}} \mathcal{M}_k(Np^e, \chi; A), \quad \mathcal{S}_k^{\mathrm{ord}}(N, \chi; A) = e_{\mathrm{ord}} \mathcal{S}_k(Np^e, \chi; A)$$

where  $e$  is any integer that is greater than the exponent of the  $p$ -primary part of the conductor of  $\chi$ . A key result in Hida's theory for ordinary  $\mathbf{I}$ -adic cusp forms is that if  $\mathbf{f} \in \mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I})$ , then for every arithmetic point  $Q \in \mathfrak{X}_{\mathbf{I}}^+$ , we have  $\mathbf{f}_Q \in \mathcal{S}_{k_Q}^{\text{ord}}(N, \chi \omega^{-k_Q} \epsilon_Q; \mathcal{O}(Q))$ . We call  $\mathbf{f} \in \mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I})$  a *primitive Hida family* if  $\mathbf{f}_Q$  is a cuspidal  $p$ -stabilized newform of tame level  $N$  for every arithmetic point  $Q \in \mathfrak{X}_{\mathbf{I}}^+$ .

## 6. A $p$ -ADIC FAMILY OF PULL-BACKS OF SIEGEL EISENSTEIN SERIES

**6.1. Siegel Eisenstein series.** We work in adèlic form, which allows us to assemble Eisenstein series out of local data. Put  $K_n = \text{U}(n) \text{GSp}_{2n}(\widehat{\mathbf{Z}})$ . Fix characters  $\chi, \hat{\omega}$  of  $\mathbf{Z}_p^\times$  of finite order and extend them to Hecke characters  $\chi_{\mathbf{A}}, \hat{\omega}_{\mathbf{A}} : \mathbf{Q}^\times \backslash \mathbf{A}^\times \rightarrow \mathbf{C}^\times$  by composition with the quotient map  $\mathbf{Q}^\times \backslash \mathbf{R}_+ \backslash \mathbf{A}^\times \simeq \widehat{\mathbf{Z}}^\times \rightarrow \mathbf{Z}_p^\times$ . We regard  $\chi$  as either a  $p$ -adic character or a complex character via composition with  $\iota_p$ . For each place  $v$  we write  $\chi_v$  for the restriction of  $\chi_{\mathbf{A}}$  to  $\mathbf{Q}_v^\times$ . Our setting means that  $\chi_p = \chi$  and  $\chi_\ell(\ell) = \chi(\ell)^{-1}$  for  $\ell \neq p$ . Let

$$I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}} \hat{\omega}_{\mathbf{A}} \alpha_{\mathbf{A}}^s) = \text{Ind}_{\mathcal{P}_3(\mathbf{A})}^{\text{GSp}_6(\mathbf{A})} (\chi_{\mathbf{A}}^2 \hat{\omega}_{\mathbf{A}} \boxtimes \chi_{\mathbf{A}}^{-3} \hat{\omega}_{\mathbf{A}}^{-1} \alpha_{\mathbf{A}}^s) \simeq \otimes'_v I_3(\hat{\omega}_v^{-1}, \chi_v \hat{\omega}_v \alpha_{\mathbf{Q}_v}^s)$$

be the global degenerate principal series representation of  $\text{GSp}_6(\mathbf{A})$  on the space of right  $K_3$ -finite functions  $f : \text{GSp}_6(\mathbf{A}) \rightarrow \mathbf{C}$  satisfying the transformation laws

$$f(\mathbf{n}(z) \mathbf{m}(A, \nu) g) = \hat{\omega}_{\mathbf{A}}(\nu^{-1} \det A) \chi_{\mathbf{A}}(\nu^{-3} (\det A)^2) |\nu^{-3} (\det A)^2|_{\mathbf{A}}^{1+s} f(g)$$

for  $A \in \text{GL}_3(\mathbf{A})$ ,  $\nu \in \mathbf{A}^\times$ ,  $z \in \text{Sym}_3(\mathbf{A})$  and  $g \in \text{GSp}_6(\mathbf{A})$ . We define global holomorphic sections of  $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}} \hat{\omega}_{\mathbf{A}} \alpha_{\mathbf{A}}^s)$  similarly. The Eisenstein series associated to a holomorphic section  $f_s$  of  $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}} \hat{\omega}_{\mathbf{A}} \alpha_{\mathbf{A}}^s)$  is defined by

$$E_{\mathbf{A}}(g, f_s) = \sum_{\gamma \in \mathcal{P}_3(\mathbf{Q}) \backslash \text{GSp}_6(\mathbf{Q})} f_s(\gamma g).$$

Such series is absolutely convergent for  $\text{Re } s > 1$  and can be continued to a meromorphic function in  $s$  on the whole plane.

Let  $k$  be an integer and  $\lambda$  a parity type. Fix a square-free integer  $N$  which is not divisible by  $p$ . We write  $\hat{\omega} = \omega_1 \omega_2 \omega_3$  as a product of three characters  $\omega_1, \omega_2, \omega_3$  of  $\mathbf{Z}_p^\times$ . Set

$$\mathcal{D} = (\chi, \omega_1, \omega_2, \omega_3).$$

Assume that  $\hat{\omega}_\infty = \text{sgn}^{k-\lambda_1}$ . Now we define a holomorphic section of  $I_3(\hat{\omega}_v^{-1}, \chi_v \hat{\omega}_v \alpha_{\mathbf{Q}_v}^s)$  for  $v \nmid N$ :

- In the archimedean case we consider the section  $f_{s, \infty}^{[k, \lambda]}$  defined in §4.1;
- In the  $p$ -adic case we consider  $f_{\mathcal{D}, s, p}$ , where the section  $f_{\mathcal{D}, s, p}$  of  $I_3(\hat{\omega}_p^{-1}, \chi_p \hat{\omega}_p \alpha_{\mathbf{Q}_p}^s)$  is attached to the quadruplet  $\mathcal{D}$  in Definition 2.5;
- If  $\ell$  and  $Np$  are coprime, then  $f_{s, \ell}^0$  is the section with  $f_{s, \ell}^0(\text{GSp}_6(\mathbf{Z}_\ell)) = 1$ .

Let  $f_{s, N}$  be an arbitrary holomorphic section of  $\otimes_{\ell|N} I_3(\hat{\omega}_\ell^{-1}, \chi_\ell \hat{\omega}_\ell \alpha_{\mathbf{Q}_\ell}^s)$  for the moment. We define the normalized Siegel Eisenstein series

$$E_{\mathbf{A}}^*(g, f_{\mathcal{D}, s, N}^{[k, \lambda]}) = L^{(\infty p N)}(2s + 2, \chi_{\mathbf{A}}^2 \hat{\omega}_{\mathbf{A}}) L^{(\infty p N)}(4s + 2, \chi_{\mathbf{A}}^4 \hat{\omega}_{\mathbf{A}}^2) \gamma_{(k, l, m)}^*(s)^{-1} \cdot E_{\mathbf{A}}(g, f_{\mathcal{D}, s, N}^{[k, \lambda]}),$$

where  $\gamma_{(k, l, m)}^*(s)$  is defined in §4.4 and  $f_{\mathcal{D}, s, N}^{[k, \lambda]}$  is a global holomorphic section of  $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}} \hat{\omega}_{\mathbf{A}} \alpha_{\mathbf{A}}^s)$  defined by

$$f_{\mathcal{D}, s, N}^{[k, \lambda]}(g) = f_{s, \infty}^{[k, \lambda]}(g_\infty) f_{s, N}((g_\ell)_{\ell|N}) f_{\mathcal{D}, s, p}(g_p) \prod_{\ell \nmid Np} f_{s, \ell}^0(g_\ell).$$

Since  $f_{\mathcal{D}, s, p}$  is supported in the big cell  $\mathcal{P}_3(\mathbf{Q}_p) J_3 \mathcal{P}_3(\mathbf{Q}_p)$ , we have the Fourier expansion

$$(6.1) \quad E_{\mathbf{A}}(g, f_{\mathcal{D}, s, N}^{[k, \lambda]}) = \sum_{B \in \text{Sym}_3(\mathbf{Q})} W_B(g, f_{\mathcal{D}, s, N}^{[k, \lambda]})$$

if  $g_p \in \mathcal{P}_3(\mathbf{Q}_p)$ . Recall that for a holomorphic section  $f_s$  of  $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}} \hat{\omega}_{\mathbf{A}} \alpha_{\mathbf{A}}^s)$

$$W_B(g, f_s) = \int_{\text{Sym}_3(\mathbf{A})} f_s(J_3 \mathbf{n}(z) g) \psi_{\mathbf{Q}}(-\text{tr}(Bz)) dz.$$

**6.2. The Fourier expansion of the pull-back of Eisenstein series.** Recall that

$$\Xi_p = \{(b_{ij}) \in \text{Sym}_3(\mathbf{Z}_p) \mid b_{11}, b_{22}, b_{33} \in p\mathbf{Z}_p \text{ and } b_{12}, b_{23}, b_{31} \in \mathbf{Z}_p^\times\}.$$

Now we evaluate its pull-back at  $s_0 = \frac{k-\lambda_1}{2} - r - 1$ . Let  $\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(f_{s_0,N}): \mathfrak{H}_1^3 \rightarrow \mathbf{C}$  be the modular form of weight  $(k, k - \lambda_2, k - \lambda_3)$  defined by

$$\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(x + y\sqrt{-1}, f_{s_0,N}) := \lim_{s \rightarrow s_0} \frac{E_{\mathbf{A}}^*(\iota(\mathbf{n}(x_1)\mathbf{m}(\sqrt{y_1}), \mathbf{n}(x_2)\mathbf{m}(\sqrt{y_2}), \mathbf{n}(x_3)\mathbf{m}(\sqrt{y_3})), f_{\mathcal{D},s,N}^{[k,\lambda]})}{\sqrt{y_1}^k \sqrt{y_2}^{k-\lambda_2} \sqrt{y_3}^{k-\lambda_3}}$$

for  $y = (y_1, y_2, y_3) \in \mathbf{R}_+^3$  and  $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ .

Since  $\omega_i$  factors through the quotient  $\mathbf{Z}_p^\times \rightarrow (\mathbf{Z}/p^{c(\omega_i)}\mathbf{Z})^\times$ , we can view  $\omega_i$  as a Dirichlet character. The polynomial  $F_{B,\ell}$  is defined in §2.2. We here set  $\mathbf{Q}_N = \prod_{\ell|N} \mathbf{Q}_\ell$ . Let  $\text{Sym}_3^+$  denote the set of positive definite rational symmetric matrices of rank 3.

**Proposition 6.1.** *Put  $n = \max\{1, c(\chi), c(\omega_i)\}$ . The pull-back  $\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(f_{s_0,N})$  is a nearly holomorphic cusp form on  $\mathfrak{H}_1^3$  of level  $\Gamma_0(Np^{2n})^3$  and nebentypus  $(\omega_1^{-1}, \omega_2^{-1}, \omega_3^{-1})$  with Fourier expansion given by*

$$\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(f_{s_0,N}) = \frac{C_1^{[k,r,\lambda]}}{\gamma_{(k,l,m)}^*(\frac{k-\lambda_1}{2} - r - 1)} \sum_{B \in \text{Sym}_3^+ \cap \Xi_p} \mathbf{W}_B^{[k,r,\lambda]}(y) \cdot \mathcal{Q}_B(\mathcal{D}) a_B(\chi^2 \hat{\omega}, k - 2r - \lambda_1) b_{B,N}^{[k,r,\lambda]} q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}},$$

where

$$a_B(\chi^2 \hat{\omega}, k - 2r - \lambda_1) := \prod_{\ell|Np} F_{B,\ell}(\chi_\ell(\ell)^2 \hat{\omega}_\ell(\ell) \ell^{2r+\lambda_1-k}),$$

$$b_{B,N}^{[k,r,\lambda]} := \lim_{s \rightarrow \frac{k-\lambda_1}{2} - r - 1} \int_{\text{Sym}_3(\mathbf{Q}_N)} f_{s,N}(J_3 \mathbf{n}(z)) \psi_{\mathbf{Q}}(-\text{tr}(Bz)) dz.$$

**PROOF.** The level and nebentypus are determined by Proposition 2.6. Note that  $\det B \in \mathbf{Z}_p^\times$  for  $B \in \Xi_p$ . In particular,  $W_B(g, f_{\mathcal{D},s,N}^{[k,\lambda]}) = 0$  unless  $\det B \neq 0$ . Lemma 4.4 says that  $W_B(g, f_{\mathcal{D},s,N}^{[k,\lambda]}) = 0$  unless  $B \in T_3^+$ . We can derive the Fourier expansion formula from (6.1), recalling that local Whittaker functions

$$\lim_{s \rightarrow \frac{k-\lambda_1}{2} - r - 1} W_B(\mathbf{m}(A), f_{s,\infty}^{[k,\lambda]}), \quad W_B(\mathbf{1}_6, f_{s,\ell}^0), \quad W_B(\mathbf{1}_6, f_{\mathcal{D},s,p})$$

are computed in (2.3), Proposition 2.6 and Lemma 4.4, respectively.  $\square$

**6.3. Holomorphic and ordinary projections of  $\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}$ .** Recall that  $\lambda_z$  is the weight-lowering operator defined in §5.2. We write  $\text{Hol}$  for the holomorphic projection. Let  $T_3^+$  denote the set of positive definite symmetric half-integral matrices of rank 3.

**Definition 6.2.** Define a holomorphic section  $f_{s,\ell}$  of  $I_3(\hat{\omega}_\ell^{-1}, \chi_\ell \hat{\omega}_\ell \alpha_{\mathbf{Q}_\ell}^s)$  by letting  $f_{s,\ell} = f_{\Phi_\ell,s}$  with  $\Phi_\ell = \mathbb{I}_{\text{Sym}_3(\mathbf{Z}_\ell)}$ . When  $f_{s,N} = \otimes_{\ell|N} f_{s,\ell}$ , we write  $\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]} = \mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(f_{s_0,N})$ . If  $B \in \text{Sym}_3^+$ , then  $b_{B,N}^{[k,r,\lambda]} = 1$  by (2.12). Proposition 6.1 gives

$$\mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]} = \frac{C_1^{[k,r,\lambda]}}{\gamma_{(k,l,m)}^*(\frac{k-\lambda_1}{2} - r - 1)} \sum_{B \in T_3^+ \cap \Xi_p} \mathbf{W}_B^{[k,r,\lambda]}(y) \cdot \mathcal{Q}_B(\mathcal{D}) a_B(\chi^2 \hat{\omega}, k - 2r - \lambda_1) q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}}.$$

**Proposition 6.3.** *Let  $\lambda$  be the parity type of  $(k, l, m)$  and  $r$  an integer which satisfies*

$$k - \frac{l+m+\lambda_1}{2} \leq r \leq \frac{l+m}{2} - 2.$$

*Put  $n = k - r - 2 + \frac{l+m-\lambda_1}{2}$ . Then  $e_{\text{ord}} \text{Hol} \left( \lambda_{z_2}^{\frac{k-l-\lambda_2}{2}} \lambda_{z_3}^{\frac{k-m-\lambda_3}{2}} \mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]} \right)$  has the  $q$ -expansion*

$$(-1)^{k+\frac{m+l+\lambda_1}{2}+\lambda_2} \sum_{B=(b_{ij}) \in T_3^+ \cap \Xi_p} \mathcal{Q}_B(\chi \varepsilon_{\text{cyc}}^n, \omega_1 \varepsilon_{\text{cyc}}^{-k}, \omega_2 \varepsilon_{\text{cyc}}^{-l}, \omega_3 \varepsilon_{\text{cyc}}^{-m}) a_B(\chi^2 \hat{\omega}, k - 2r - \lambda_1) q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}}.$$

PROOF. Put  $b = \frac{k-l-\lambda_2}{2}$  and  $c = \frac{k-m-\lambda_3}{2}$ . If  $f$  is a holomorphic function on  $\mathfrak{H}_1$ , then

$$\lambda_z^n (y^{-a} f) = \begin{cases} (4\pi)^{-n} n! \binom{a}{n} \cdot y^{n-a} f & \text{if } n \leq a, \\ 0 & \text{if } n > a. \end{cases}$$

By (4.4) the difference

$$\lambda_{z_2}^b \lambda_{z_3}^c \mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(q) - C_1^{[k,r,\lambda]} \frac{b!c!}{(4\pi)^{b+c}} \sum_B Q_{0,b,c}^{[k,\lambda]}(B,r) \mathcal{Q}_B(\mathcal{D})_{a_B}(\chi^2 \hat{\omega}, k-2r-\lambda_1) q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}}$$

belongs to  $(y_1^{-1}, y_2^{-1}, y_3^{-1}) \mathbf{C}[y_1^{-1}, y_2^{-1}, y_3^{-1}][[q_1, q_2, q_3]]$ . On the other hand, we can write

$$\lambda_{z_2}^b \lambda_{z_3}^c \mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]}(q) = \text{Hol}(\lambda_{z_2}^b \lambda_{z_3}^c \mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]})(q) + \sum_{i+j+t \geq 1} \delta_{k-i}^i f_i(q_1) \delta_{l-j}^j g_j(q_2) \delta_{m-t}^t h_t(q_3),$$

where  $f_i, g_j$  and  $h_t$  are holomorphic modular forms. Equating the constant terms of this identity as a polynomial in  $y_1^{-1}, y_2^{-1}, y_3^{-1}$  and employing the relation

$$(6.2) \quad \delta_k^t = \sum_{a=0}^t \binom{t}{a} \frac{\Gamma(t+k)}{\Gamma(a+k)} (-4\pi y)^{a-t} \left( \frac{1}{2\pi\sqrt{-1}} \frac{\partial}{\partial z} \right)^a$$

(see [Hid93, (3), page 311]), we see that the holomorphic projection  $\text{Hol}(\lambda_{z_2}^b \lambda_{z_3}^c \mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]})(q)$  equals

$$C_1^{[k,r,\lambda]} \frac{b!c!}{(4\pi)^{b+c}} \sum_B Q_{0,b,c}^{[k,\lambda]}(B,r) \mathcal{Q}_B(\mathcal{D})_{a_B}(\chi^2 \hat{\omega}, k-2r-\lambda_1) q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}} - \sum_{i+j+t \geq 1} \theta^i f_i(q_1) \theta^j g_j(q_2) \theta^t h_t(q_3).$$

Here  $\theta$  stands for the Serre's operator  $\theta(\sum_i a_i q^i) = \sum_i i a_i q^i$ . Since  $e_{\text{ord}} \theta = 0$ , the  $q$ -expansion of the ordinary projection  $e_{\text{ord}} \text{Hol}(\lambda_{z_2}^b \lambda_{z_3}^c \mathbf{E}_{\mathcal{D},N}^{[k,r,\lambda]})(q)$  equals

$$C_1^{[k,r,\lambda]} \frac{b!c!}{(4\pi)^{b+c}} \sum_B \mathbf{c}_B \cdot q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}},$$

where

$$\mathbf{c}_B = \lim_{j \rightarrow \infty} Q_{0,b,c}^{[k,\lambda]}(B_j, r) \mathcal{Q}_{B_j}(\mathcal{D})_{a_{B_j}}(\chi^2 \hat{\omega}, k-2r-\lambda_1), \quad B_j := \begin{pmatrix} p^{j^1} b_{11} & b_{12} & b_{13} \\ b_{12} & p^{j^1} b_{22} & b_{23} \\ b_{13} & b_{23} & p^{j^1} b_{33} \end{pmatrix}.$$

Since  $p^{j^1} \rightarrow 1$  in  $\mathbf{Z}_\ell$  as  $j \rightarrow \infty$  for any rational prime  $\ell \neq p$ , we get

$$\mathcal{Q}_{B_j}(\mathcal{D}) = \mathcal{Q}_B(\mathcal{D}), \quad \lim_{j \rightarrow \infty} a_{B_j}(\chi^2 \hat{\omega}, k-2r-\lambda_1) = a_B(\chi^2 \hat{\omega}, k-2r-\lambda_1).$$

Since  $Q_{0,b,c}^{[k,\lambda]}(B,r)$  is a polynomial in  $B$ , we find that

$$\mathbf{c}_B = Q_{0,b,c}^{[k,\lambda]}(B_\infty, r) \mathcal{Q}_B(\mathcal{D})_{a_B}(\chi^2 \hat{\omega}, k-2r-\lambda_1), \quad B_\infty = \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{12} & 0 & b_{23} \\ b_{13} & b_{23} & 0 \end{pmatrix}.$$

Applying Lemma 4.7 to  $Q_{0,b,c}^{[k,\lambda]}(B_\infty, r)$ , we obtain

$$\mathbf{c}_B = w_{0,b,c} \cdot 2^{-3n+k+l+m} \mathcal{Q}_B(\chi \varepsilon_{\text{cyc}}^n, \omega_1 \varepsilon_{\text{cyc}}^{-k}, \omega_2 \varepsilon_{\text{cyc}}^{-l}, \omega_3 \varepsilon_{\text{cyc}}^{-m}) a_B(\chi^2 \hat{\omega}, k-2r-\lambda_1)$$

in view of Definition 2.5 of  $\mathcal{Q}_B$ . We thus obtain the lemma by noting the equality

$$(-1)^{k+\frac{m+l+\lambda_1}{2}+\lambda_2} \gamma_{(k,l,m)}^* \left( \frac{k-\lambda_1}{2} - r - 1 \right) = C_1^{[k,r,\lambda]} b!c! (4\pi)^{-b-c} 2^{-3n+k+l+m} \cdot w_{0,b,c}.$$

The constant  $C_1^{[k,r,\lambda]}$  is defined in Lemma 4.4. The equality can be checked by the following items:

- The power of 2:

$$\begin{aligned} & 3(3+2r-k-\lambda_2) + \{2(k-r)-3\} - 2b-2c + (k+l+m-3n) \\ & + (7M-3b-3c-3\lambda_1) = -2-k+2(l+m)+\lambda_1+2\lambda_2. \end{aligned}$$

- The power of  $\pi$ :  $(6-2) - b - c + (3M - b - c - 2\lambda_1 - \lambda_2) = -3r + k + l + m + \lambda_2 - \lambda_1 - 2$ .

□

#### 6.4. The modular forms $G_{k_1, k_2, k_3}^{[n]}(\mathcal{D})$ .

**Definition 6.4.** Let  $(k_1, k_2, k_3)$  be a triplet of positive integers. Put  $k^* = \max\{k_1, k_2, k_3\}$ . We say that  $(k_1, k_2, k_3)$  is *balanced* if  $2k^* < k_1 + k_2 + k_3$ . An integer  $n$  is said to be *critical* for  $(k_1, k_2, k_3)$  if

$$k^* \leq n \leq k_1 + k_2 + k_3 - k^* - 2.$$

**Definition 6.5.** Fix a balanced triplet  $(k_1, k_2, k_3)$  of positive integers. Take a permutation  $\sigma$  of  $\{1, 2, 3\}$  so that  $k^* = k_{\sigma(1)} \geq k_{\sigma(2)} \geq k_{\sigma(3)}$ . Denote the parity type of  $(k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)})$  by  $\delta = (\delta_1, \delta_2, \delta_3)$ . For each critical integer  $n$  for  $(k_1, k_2, k_3)$  and quadruplet  $\mathcal{D} = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$  of finite-order  $p$ -adic characters of  $\mathbf{Z}_p^\times$  we define the modular form  $G_{k_1, k_2, k_3}^{[n]}(\mathcal{D})$  by

$$G_{k_1, k_2, k_3}^{[n]}(\mathcal{D}) := (-1)^{k + \frac{m+l+\lambda_1}{2} + \lambda_2} e_{\text{ord}} \text{Hol} \left( \lambda_{z_{\sigma(2)}}^{\frac{k^* - k_{\sigma(2)} - \delta_2}{2}} \lambda_{z_{\sigma(3)}}^{\frac{k^* - k_{\sigma(3)} - \delta_3}{2}} \mathbf{E}_{\mathcal{D}}^{[k^*, r, \delta]} \right),$$

where  $r = \left\lfloor \frac{k^* + k_1 + k_2 + k_3}{2} \right\rfloor - n - 2$  and  $\mathcal{D} = (\iota_p \circ \epsilon_0, \iota_p \circ \epsilon_1, \iota_p \circ \epsilon_2, \iota_p \circ \epsilon_3)$ .

**Corollary 6.6.** *With notation in Definition 6.5,  $G_{k_1, k_2, k_3}^{[n]}(\mathcal{D})$  is an ordinary cusp form of weight  $(k_1, k_2, k_3)$ , level  $\Gamma_0(Np^\infty)^3$  and nebentypus  $(\epsilon_1^{-1}, \epsilon_2^{-1}, \epsilon_3^{-1})$  whose  $q$ -expansion at the infinity cusp is given by*

$$\sum_{B=(b_{ij}) \in T_3^+ \cap \Xi_p} \mathcal{Q}_B(\epsilon_0 \epsilon_{\text{cyc}}^n, \epsilon_1 \epsilon_{\text{cyc}}^{-k_1}, \epsilon_2 \epsilon_{\text{cyc}}^{-k_2}, \epsilon_3 \epsilon_{\text{cyc}}^{-k_3}) a_B(\epsilon_0^2 \epsilon_1 \epsilon_2 \epsilon_3, 2n - (k_1 + k_2 + k_3) + 4) \cdot q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}}.$$

**PROOF.** The assertion for the Fourier expansion is a direct consequence of Proposition 6.3 by symmetry. Lemma 6.7 below implies the cuspidality of  $G_{k_1, k_2, k_3}^{[n]}(\mathcal{D})$ . □

**Lemma 6.7.** *Let  $f \in \mathcal{M}_k^{\text{ord}}(N, \chi; A)$ . Assume that  $\mathbf{a}_0(g, \Phi(f)) = 0$  whenever  $g_p \in B_2(\mathbf{Q}_p)$ . Then  $f \in \mathcal{S}_k^{\text{ord}}(N, \chi; A)$ .*

**PROOF.** Our task is to prove that  $\mathbf{a}_0(g, \Phi(f)) = 0$  for all  $g \in \text{GL}_2(\mathbf{A})$ . Since

$$\mathbf{a}_0(\gamma \mathbf{n}(x) \text{diag}(a, d) g \kappa_\theta, \Phi(f)) = (ad^{-1})^{k/2} e^{\sqrt{-1}k\theta} \mathbf{a}_0(g_{\mathbf{f}}, \Phi(f))$$

for  $\gamma \in B_2(\mathbf{Q})$ ,  $x \in \mathbf{A}$ ,  $a, d \in \mathbf{R}_+$  and  $\theta \in \mathbf{R}$ , it suffices to show that  $\mathbf{a}_0(g, \Phi(f)) = 0$  for all  $g \in \text{GL}_2(\widehat{\mathbf{Z}})$ . Since

$$\text{GL}_2(\mathbf{Z}_p) = \mathbf{n}^-(p\mathbf{Z}_p)B_2(\mathbf{Z}_p) \sqcup \mathbf{n}(\mathbf{Z}_p)J_1B_2(\mathbf{Z}_p),$$

where  $J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , we have only to show that  $\mathbf{a}_0(h\mathbf{n}^-(y), \Phi(f)) = \mathbf{a}_0(hJ_1, \Phi(f)) = 0$  for all  $h \in \text{GL}_2(\widehat{\mathbf{Z}}^{(p)})$  and  $y \in p\mathbf{Z}_p$ . Recall that the operator  $\mathbf{U}_p$  is defined by

$$[\mathbf{U}_p \Phi](g, f) = p^{(k-2)/2} \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} \Phi \left( g \begin{pmatrix} \varpi_p & x \\ 0 & 1 \end{pmatrix}, f \right).$$

Recall that  $\varpi_p \in \widehat{\mathbf{Q}}^\times$  is defined by  $\varpi_{p,p} = p$  and  $\varpi_{p,\ell} = 1$  for  $\ell \neq p$ . Since

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} \varpi_p^m & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \varpi_p^m & x \\ 1+xy & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi_p^m y & 1+xy \end{pmatrix} \in B_2(\mathbf{Q}_p)U_0(N)$$

for  $y \in p\mathbf{Z}_p$ ,  $x \in \mathbf{Z}_p$  and sufficiently large  $m$ , we get

$$\mathbf{a}_0(h\mathbf{n}^-(y), \mathbf{U}_p^m f) = p^{(k-2)m/2} \sum_{x \in \mathbf{Z}_p/p\mathbf{Z}_p} \Phi \left( h \begin{pmatrix} \varpi_p^m & x \\ 1+xy & 1 \end{pmatrix}, f \right) = 0$$

by assumption. It follows that  $\mathbf{a}_0(h\mathbf{n}^-(y), f) = \lim_{n \rightarrow \infty} \mathbf{a}_0(h\mathbf{n}^-(y), \mathbf{U}_p^n f) = 0$ . If  $x \in p^n \mathbf{Z}_p^\times$  with  $n < m$ , then

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varpi_p^m & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \varpi_p^{m-n} & -\varpi_p^n x^{-1} \\ 0 & \varpi_p^n \end{pmatrix} \begin{pmatrix} \varpi_p^n x^{-1} & 0 \\ \varpi_p^{m-n} & \varpi_p^{-n} x \end{pmatrix} \in B_2(\mathbf{Q}_p) \mathbf{n}^-(p\mathbf{Z}_p).$$

One can therefore see that

$$\mathbf{a}_0(hJ_1, \mathbf{U}_p^m f) = p^{(k-2)m/2} \mathbf{a}_0(hJ_1 \text{diag}(\varpi_p^m, 1), f) = p^{(k-1)m/2} \mathbf{a}_0(\text{diag}(1, \varpi_p^{-m})hJ_1, f),$$

from which we conclude that

$$\mathbf{a}_0(hJ_1, f) = \lim_{n \rightarrow \infty} p^{(k-1)n/2} \mathbf{a}_0(\text{diag}(1, \varpi_p^{-n})hJ_1, f) = 0.$$

Here  $\varpi_{(p)} \in \widehat{\mathbf{Z}}_p^\times$  is defined by  $\varpi_{(p),p} = 1$  and  $\varpi_{(p),\ell} = p$  for  $\ell \neq p$ .  $\square$

**6.5. The  $p$ -adic interpolation of  $G_{k_1, k_2, k_3}^{[n]}(\mathcal{D})$ .** We give the construction the  $p$ -adic triple  $L$ -function in this subsection. Let  $\mathbf{u} = 1 + p \in 1 + p\mathbf{Z}_p$  be a topological generator. We identify  $\mathcal{O}[\text{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]$  with  $\mathcal{O}[[X]]$  where  $X = [\mathbf{u}] - 1$  with the group-like element  $[\mathbf{u}]$  in  $\Lambda$ . Put

$$\Lambda = \mathcal{O}[\text{Gal}(\mathbf{Q}_\infty/\mathbf{Q})], \quad \Lambda_3 = \mathcal{O}[[X_1, X_2, X_3]], \quad \Lambda_4 = \Lambda_3[[T]].$$

For each  $\ell \nmid Np$  and  $B \in T_3^+$ , let  $F_{B,\ell}(X) \in \mathbf{Z}[X]$  be as defined in (2.3). Let  $\alpha_X : \mathbf{Z}_p^\times \rightarrow \mathbf{Z}_p[[X]]^\times$  be the character  $\alpha_X(z) = \langle z \rangle_X = (1+X)^{\log_p z / \log_p \mathbf{u}}$ . Let  $\underline{\chi} = (\chi_1, \chi_2, \chi_3)$  be a triplet of  $\mathcal{O}$ -valued finite-order characters of  $\mathbf{Z}_p^\times$ . For each  $a \in \mathbf{Z}/(p-1)\mathbf{Z}$  we define the formal power series  $\mathcal{G}_{\underline{\chi}}^{(a)} \in \Lambda_4[[q_1, q_2, q_3]]$  by

$$\mathcal{G}_{\underline{\chi}}^{(a)}(X_1, X_2, X_3, T) = \sum_{B=(b_{ij}) \in T_3^+ \cap \Xi_p} \mathcal{Q}_B^{(a)}(X_1, X_2, X_3, T) \cdot \mathcal{F}_B^{(a)}(X_1, X_2, X_3, T) \cdot q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}},$$

where  $\mathcal{Q}_B^{(a)}$  and  $\mathcal{F}_B^{(a)} \in \Lambda_3[[T]]$  are power series given by

$$\begin{aligned} \mathcal{Q}_B^{(a)}(X_1, X_2, X_3, T) &= \boldsymbol{\omega}(8b_{23}b_{31}b_{12})^a \langle 8b_{23}b_{31}b_{12} \rangle_T \chi_1(2b_{23})^{-1} \langle 2b_{23} \rangle_{X_1}^{-1} \chi_2(2b_{31})^{-1} \langle 2b_{31} \rangle_{X_2}^{-1} \chi_3(2b_{12})^{-1} \langle 2b_{12} \rangle_{X_3}^{-1}, \\ \mathcal{F}_B^{(a)}(X_1, X_2, X_3, T) &= \prod_{\ell \nmid pN} F_{B,\ell}(\langle \ell \rangle_{X_1, X_2, X_3, T}^{(a)} \ell^{-2}), \end{aligned}$$

where

$$\langle \ell \rangle_{X_1, X_2, X_3, T}^{(a)} := (\boldsymbol{\omega}^{-2a} \chi_1 \chi_2 \chi_3)(\ell) \ell^{-2} \cdot \langle \ell \rangle_{X_1} \langle \ell \rangle_{X_2} \langle \ell \rangle_{X_3} \langle \ell \rangle_T^{-2} \in \Lambda_4^\times.$$

Here the set  $\mathfrak{X}_{\Lambda_4}^{\text{bal}}$  consists of  $(Q, P) = (Q_1, Q_2, Q_3, P) \in (\mathfrak{X}_{\Lambda}^+)^3 \times \mathfrak{X}_{\Lambda} \subset \text{Spec } \Lambda_4(\overline{\mathbf{Q}}_p)$  such that  $(k_{Q_1}, k_{Q_2}, k_{Q_3})$  is balanced and  $k_P$  is critical for  $(k_{Q_1}, k_{Q_2}, k_{Q_3})$ .

**Proposition 6.8.** *For every  $(Q, P) \in \mathfrak{X}_{\Lambda_4}^{\text{bal}}$ , we have*

$$\mathcal{G}_{\underline{\chi}}^{(a)}(Q, P) = G_{k_{Q_1}, k_{Q_2}, k_{Q_3}}^{[k_P]}(\epsilon_P \boldsymbol{\omega}^{a-k_P}, \chi_1^{-1} \epsilon_{Q_1}^{-1} \boldsymbol{\omega}^{k_{Q_1}}, \chi_2^{-1} \epsilon_{Q_2}^{-1} \boldsymbol{\omega}^{k_{Q_2}}, \chi_3^{-1} \epsilon_{Q_3}^{-1} \boldsymbol{\omega}^{k_{Q_3}}).$$

In particular, this implies that

$$\mathcal{G}_{\underline{\chi}}^{(a)} \in \mathbf{S}^{\text{ord}}(N, \chi_1, \mathcal{O}[[X_1]]) \widehat{\otimes}_{\mathcal{O}} \mathbf{S}^{\text{ord}}(N, \chi_2, \mathcal{O}[[X_2]]) \widehat{\otimes}_{\mathcal{O}} \mathbf{S}^{\text{ord}}(N, \chi_3, \mathcal{O}[[X_3]]) \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[T]].$$

**PROOF.** Set  $\chi := \epsilon_P \boldsymbol{\omega}^{a-k_P}$ ,  $\omega_i = \chi_i^{-1} \epsilon_{Q_i}^{-1} \boldsymbol{\omega}^{k_{Q_i}}$  and  $\hat{\omega} = \omega_1 \omega_2 \omega_3$ . One can check that

$$\begin{aligned} \mathcal{Q}_B^{(a)}(Q, P) &= \frac{(\epsilon_P \boldsymbol{\omega}^a)(8b_{12}b_{23}b_{13}) \langle 8b_{12}b_{23}b_{13} \rangle^{k_P}}{(\chi_1 \epsilon_{Q_1})(2b_{23})(\chi_2 \epsilon_{Q_2})(2b_{13})(\chi_3 \epsilon_{Q_3})(2b_{12}) \langle 2b_{23} \rangle^{k_{Q_1}} \langle 2b_{31} \rangle^{k_{Q_2}} \langle 2b_{12} \rangle^{k_{Q_3}}} \\ &= \mathcal{Q}_B(\chi \boldsymbol{\epsilon}_{\text{cyc}}^{k_P}, \omega_1 \boldsymbol{\epsilon}_{\text{cyc}}^{-k_{Q_1}}, \omega_2 \boldsymbol{\epsilon}_{\text{cyc}}^{-k_{Q_2}}, \omega_3 \boldsymbol{\epsilon}_{\text{cyc}}^{-k_{Q_3}}), \\ \langle \ell \rangle_{X_1, X_2, X_3, T}^{(a)}(Q, P) &= (\boldsymbol{\omega}^{-2a} \chi_1 \chi_2 \chi_3)(\ell) \ell^{-2} \cdot (\epsilon_{Q_1} \epsilon_{Q_2} \epsilon_{Q_3} \epsilon_P^{-2} \boldsymbol{\omega}^{2k_P - k_{Q_1} - k_{Q_2} - k_{Q_3}})(\ell)^{-1} \ell^{k_{Q_1} + k_{Q_2} + k_{Q_3} - 2k_P} \\ &= \chi_\ell^2(\ell) |\ell|^{2k_P + 2} \hat{\omega}_\ell(\ell) |\ell|^{-(k_{Q_1} + k_{Q_2} + k_{Q_3})}, \\ \mathcal{F}_B^{(a)}(Q, P) &= a_B(\chi^2 \hat{\omega}, 2k_P - (k_{Q_1} + k_{Q_2} + k_{Q_3}) + 4) \end{aligned}$$

(see Definition 2.5 of  $\mathcal{Q}_B$ ). Recall the convention that  $\chi_\ell(\ell) = \iota_p(\chi(\ell))^{-1}$  and  $\hat{\omega}_\ell(\ell) = \iota_p(\hat{\omega}(\ell))^{-1}$  (see Remark 5.2). From Corollary 6.6, we deduce the interpolation formula and that

$$(6.3) \quad \mathcal{G}_{\underline{\chi}}^{(a)}(Q, P) \in \mathbf{S}_{k_{Q_1}}^{\text{ord}}(N, \omega_1^{-1}; \mathcal{O}(Q_1)) \widehat{\otimes}_{\mathcal{O}} \mathbf{S}_{k_{Q_2}}^{\text{ord}}(N, \omega_2^{-1}; \mathcal{O}(Q_2)) \widehat{\otimes}_{\mathcal{O}} \mathbf{S}_{k_{Q_3}}^{\text{ord}}(N, \omega_3^{-1}; \mathcal{O}(Q_3)) \widehat{\otimes}_{\mathcal{O}} \mathcal{O}(P).$$

By the control theorem for ordinary  $\Lambda$ -adic forms [Hid93, Theorem 3, p.215], for any arithmetic point  $Q$ , the specialization map  $X \mapsto \mathbf{u}^{k_Q} \epsilon_Q(\mathbf{u}) - 1$  yields an isomorphism

$$\mathbf{S}^{\text{ord}}(N, \chi, \mathcal{O}[[X]]) / (1 + X - \mathbf{u}^{k_Q} \epsilon_Q(\mathbf{u})) \simeq \mathbf{S}_{k_Q}^{\text{ord}}(N, \chi \boldsymbol{\omega}^{-k_Q} \epsilon_Q; \mathcal{O}(Q)).$$



Hence, from (6.3) we find that for all  $P$  with  $k_P = 2$

$$\mathcal{G}_{\underline{\chi}}^{(a)}(X_1, X_2, X_3, P) \in \mathbf{S}^{\text{ord}}(N, \chi_1, \mathcal{O}[[X_1]]) \widehat{\otimes}_{\mathcal{O}} \mathbf{S}^{\text{ord}}(N, \chi_2, \mathcal{O}[[X_2]]) \widehat{\otimes}_{\mathcal{O}} \mathbf{S}^{\text{ord}}(N, \chi_3, \mathcal{O}[[X_3]]) \otimes_{\mathcal{O}} \mathcal{O}(P).$$

Now we can deduce the second statement from the above equation combined with the argument in [Hid93, Lemma 1, page 328].  $\square$

## 7. FOUR-VARIABLE $p$ -ADIC TRIPLE PRODUCT $L$ -FUNCTIONS

**7.1. Measures.** We shall normalize the Haar measures  $dx_v$  on  $\mathbf{Q}_v$  and  $d^\times x_v$  on  $\mathbf{Q}_v^\times$  as follows: Let  $dx_\infty$  denote the usual Lebesgue measure on  $\mathbf{R}$  and  $d^\times x_\infty = \frac{dx_\infty}{|x_\infty|}$ . If  $v = \ell$  is finite, then  $\text{vol}(\mathbf{Z}_\ell, dx_\ell) = \text{vol}(\mathbf{Z}_\ell^\times, d^\times x_\ell) = 1$ . Define the compact subgroups  $\mathbf{K}_v$  of  $\text{GL}_2(\mathbf{Q}_v)$  and  $\mathbf{K}'_v$  of  $\text{SL}_2(\mathbf{Q}_v)$  by

$$\mathbf{K}_\infty = \text{O}(2, \mathbf{R}), \quad \mathbf{K}_\ell = \text{GL}_2(\mathbf{Z}_\ell), \quad \mathbf{K}'_\infty = \text{SO}(2, \mathbf{R}), \quad \mathbf{K}'_\ell = \text{SL}_2(\mathbf{Z}_\ell).$$

Let  $dk_v$  and  $dk'_v$  be the Haar measures on  $\mathbf{K}_v$  and  $\mathbf{K}'_v$  which have total volume 1.

The Haar measure  $dg_v$  on  $\text{PGL}_2(\mathbf{Q}_v)$  is given by  $dg_v = |y_v|_v^{-1} dx_v d^\times y_v dk_v$  for  $g_v = \begin{pmatrix} y_v & x_v \\ 0 & 1 \end{pmatrix}$  with  $y_v \in \mathbf{Q}_v^\times$ ,  $x_v \in \mathbf{Q}_v$  and  $k_v \in \mathbf{K}_v$ . Define the Haar measure  $dg'_v$  on  $\text{SL}_2(\mathbf{Q}_v)$  by  $dg'_v = |y_v|_v^{-2} dx_v d^\times y_v dk'_v$  for  $g_v = \mathbf{n}(x_v)\mathbf{m}(y_v)k'_v$  with  $y_v \in \mathbf{Q}_v^\times$ ,  $x_v \in \mathbf{Q}_v$  and  $k'_v \in \mathbf{K}'_v$ . The Tamagawa measures  $dg$  on  $\text{PGL}_2(\mathbf{A})$  and  $dg'$  on  $\text{SL}_2(\mathbf{A})$  are given by  $dg = \zeta_{\mathbf{Q}}(2)^{-1} \prod_v dg_v$  and  $dg' = \zeta_{\mathbf{Q}}(2)^{-1} \prod_v dg'_v$ . Since  $Z \backslash H \simeq \text{PGL}_2 \times \text{SL}_2 \times \text{SL}_2$ , we can define the Tamagawa measure on  $Z \backslash H$  by  $dg_1 dg'_2 dg'_3$ , where  $dg_1$  is the Tamagawa measure on  $\text{PGL}_2(\mathbf{A})$  and  $dg'_2 = dg'_3$  are that on  $\text{SL}_2(\mathbf{A})$ . The Tamagawa numbers of  $\text{PGL}_2$ ,  $\text{SL}_2$  and  $Z \backslash H$  are 2, 1 and 2, respectively.

**7.2. Garrett's integral representation.** Let  $\pi_i$  ( $i = 1, 2, 3$ ) be an irreducible cuspidal automorphic representation of  $\text{GL}_2(\mathbf{A})$  generated by an elliptic cusp form of weight  $k_i$  and nebentypus  $\omega_i^{-1}$ . Put  $\hat{\omega} = \omega_1 \omega_2 \omega_3$  and  $\hat{\pi}_i = \pi_i \otimes \omega_{i, \mathbf{A}}^{-1}$  for  $i = 1, 2, 3$ . Fix a character  $\chi_{\mathbf{A}}$  of  $\mathbf{A}^\times / \mathbf{Q}^\times \mathbf{R}_+$ . For each triplet of cusp forms  $\varphi_i \in \hat{\pi}_i$  and a holomorphic section  $f_s$  of  $I_3(\hat{\omega}_{\mathbf{A}}^{-1}, \chi_{\mathbf{A}} \hat{\omega}_{\mathbf{A}} \alpha_{\mathbf{A}}^s)$  we consider the global zeta integral defined by

$$Z(\varphi_1, \varphi_2, \varphi_3, E_{\mathbf{A}}(f_s)) = \int_{Z(\mathbf{A})H(\mathbf{Q}) \backslash H(\mathbf{A})} \varphi_1(g_1) \varphi_2(g_2) \varphi_3(g_3) E_{\mathbf{A}}(\iota(g_1, g_2, g_3), f_s) dg_1 dg_2 dg_3.$$

The integral converges absolutely for all  $s$  away from the poles of the Eisenstein series and is hence meromorphic in  $s$ . Unfolding the Eisenstein series as in [PSR87], we get

$$Z(\varphi_1, \varphi_2, \varphi_3, E_{\mathbf{A}}(f_s)) = \int_{Z(\mathbf{A})U^0(\mathbf{A}) \backslash H(\mathbf{A})} W(g_1, \varphi_1) W(g_2, \varphi_2) W(g_3, \varphi_3) f_s(\delta \iota(g_1, g_2, g_3)) dg_1 dg_2 dg_3.$$

If  $W(g, \varphi_i) = \prod_v W_{i,v}(g_v)$  and  $f_s(g) = \prod_v f_{s,v}(g_v)$  are factorizable, then the integral factors into a product of local integrals and so by §2.2

$$Z(\varphi_1, \varphi_2, \varphi_3, E_{\mathbf{A}}(f_s)) = \frac{\zeta_{\mathbf{Q}}(2)^{-3} L(s + \frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})}{L^S(2s + 2, \chi_{\mathbf{A}}^2 \hat{\omega}_{\mathbf{A}}) L^S(4s + 2, \chi_{\mathbf{A}}^4 \hat{\omega}_{\mathbf{A}}^2)} \prod_{v \in S} \frac{Z(W_{1,v}, W_{2,v}, W_{3,v}, f_{s,v})}{L(s + \frac{1}{2}, \pi_{1,v} \times \pi_{2,v} \times \pi_{3,v} \otimes \chi_v)},$$

where  $S$  is a large enough set of places such that  $\pi_{i,\ell}$ ,  $W_{i,\ell}$ ,  $\chi_\ell$  and  $f_{s,\ell}$  are unramified for all  $\ell \notin S$ . The complete  $L$ -function  $L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})$  admits meromorphic continuation and a functional equation

$$L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}}) = \varepsilon(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}}) L(1 - s, \pi_1 \times \pi_2 \times \pi_3 \otimes \hat{\omega}_{\mathbf{A}}^{-1} \chi_{\mathbf{A}}^{-1}).$$

By Theorem 2.7 of [Ike92] the  $L$ -function  $L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})$  has a pole if and only if there exists an imaginary quadratic field  $E$  and characters of  $\chi_i$  of  $\mathbf{A}_E^\times / E^\times$  such that  $\chi_1 \chi_2 \chi_3 \chi^E = 1$  and such that  $\pi_i$  is induced automorphically from  $\chi_i$ , where  $\chi^E$  denotes the base change of  $\chi$  to  $E$ . Recall that  $k^* = \max\{k_1, k_2, k_3\}$ . In particular, if  $k_1 + k_2 + k_3 \geq 2k^* + 2$ , then  $L(s, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})$  is holomorphic everywhere. Let us put

$$\mathcal{J}_\infty = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{GL}_2(\mathbf{R}), \quad t_n = \begin{pmatrix} 0 & p^{-n} \\ -p^n & 0 \end{pmatrix} \in \text{GL}_2(\mathbf{Q}_p) \hookrightarrow \text{GL}_2(\mathbf{A}).$$

Let  $E_{\mathbf{A}}^*(f_{\mathcal{D}, s, N}^{[k_1, \lambda]})$  be the Eisenstein series associated with a section  $f_{s, N} = \otimes_{\ell|N} f_{s, \ell}$  of  $\otimes_{\ell|N} I_3(\hat{\omega}_\ell^{-1}, \chi_\ell \hat{\omega}_\ell \alpha_{\mathbf{Q}_\ell}^s)$  as in §6.1.

**Lemma 7.1.** *Let  $f_i \in \mathcal{S}_{k_i}(N_i, \omega_i^{-1})$  be an ordinary  $p$ -stabilized newform. Put*

$$\varphi_i = \Phi(f_i), \quad \check{\varphi}_i = \varphi_i \otimes \omega_{i,\mathbf{A}}^{-1}, \quad W(\varphi_i) = \prod_v W_{i,v}, \quad \check{W}_{i,v} = W_{i,v} \otimes \omega_{i,v}^{-1},$$

Let  $\chi$  be a character of  $\mathbf{Z}_p^\times$  of finite order. Put  $n = \{1, c(\chi), c(\omega_i)\}$ . If  $k_1 \geq k_2 \geq k_3$  and  $\lambda$  is the parity type of  $(k_1, k_2, k_3)$ , then

$$\begin{aligned} & Z\left(\rho(\mathcal{J}_\infty t_n) \check{\varphi}_1, \rho(\mathcal{J}_\infty t_n) V_+^{\frac{k_1-k_2-\lambda_2}{2}} \check{\varphi}_2, \rho(\mathcal{J}_\infty t_n) V_+^{\frac{k_1-k_3-\lambda_3}{2}} \check{\varphi}_3, E_{\mathbf{A}}^*(f_{\mathcal{D},s,N}^{[k_1,\lambda]})\right) \\ &= L^{(N)}\left(s + \frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}}\right) E_p\left(s + \frac{1}{2}, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \chi_p\right) \\ & \times \prod_{i=1}^3 \frac{\zeta_p(2)}{\zeta_p(1)} \left(\frac{\alpha_p(f_i)^2}{p^{k_i} \omega_{i,p}(p)}\right)^n \frac{\prod_{\ell|N} Z(\check{W}_{1,\ell}, \check{W}_{2,\ell}, \check{W}_{3,\ell}, f_{s,\ell})}{\zeta_{\mathbf{Q}}(2)^{3 \cdot 2^5 + (k_1+k_2+k_3)}} \end{aligned}$$

PROOF. By Garrett's integral representation of triple  $L$ -functions the left hand side equals

$$\begin{aligned} & \zeta_{\mathbf{Q}}(2)^{-3} L^{(\infty p N)}\left(s + \frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}}\right) \\ & \times \gamma_{(k_1, k_2, k_3)}^*(s)^{-1} Z\left(\rho(\mathcal{J}_\infty) \check{W}_{1,\infty}, \rho(\mathcal{J}_\infty) V_+^{\frac{k_1-k_2-\lambda_2}{2}} \check{W}_{2,\infty}, \rho(\mathcal{J}_\infty) V_+^{\frac{k_1-k_3-\lambda_3}{2}} \check{W}_{3,\infty}, f_{s,\infty}^{[k_1,\lambda]}\right) \\ & \times Z(\rho(t_n) \check{W}_{1,p}, \rho(t_n) \check{W}_{2,p}, \rho(t_n) \check{W}_{3,p}, f_{\mathcal{D},s,p}) \prod_{\ell|N} Z(\check{W}_{1,\ell}, \check{W}_{2,\ell}, \check{W}_{3,\ell}, f_{s,\ell}) \end{aligned}$$

in view of Definition 6.2 of  $E_{\mathbf{A}}^*(f_{\mathcal{D},s}^{[k_1,\lambda]})$ . Since  $\rho(\mathcal{J}_\infty) \check{W}_{i,\infty} = \omega_i(-1) \rho(\mathcal{J}_\infty) W_{i,\infty}$ , Lemma 4.8 yields

$$\begin{aligned} & Z\left(\rho(\mathcal{J}_\infty) \check{W}_{1,\infty}, \rho(\mathcal{J}_\infty) V_+^{\frac{k_1-k_2-\lambda_2}{2}} \check{W}_{2,\infty}, \rho(\mathcal{J}_\infty) V_+^{\frac{k_1-k_3-\lambda_3}{2}} \check{W}_{3,\infty}, f_{s,\infty}^{[k_1,\lambda]}\right) \\ &= \hat{\omega}_\infty(-1) Z_\infty(s) = \chi_\infty(-1) \frac{\text{vol}(\text{SO}(2))^3 \gamma_{(k_1, k_2, k_3)}^*(s)}{2^{5+(k_1+k_2+k_3)}} L\left(s + \frac{1}{2}, \sigma_{k_1} \times \sigma_{k_2} \times \sigma_{k_3} \otimes \chi_\infty\right). \end{aligned}$$

Proposition 2.6 calculates the  $p$ -adic part:

$$\begin{aligned} & \frac{Z(\rho(t_n) \check{W}_{1,p}, \rho(t_n) \check{W}_{2,p}, \rho(t_n) \check{W}_{3,p}, f_{\mathcal{D},s,p})}{L\left(s + \frac{1}{2}, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \chi_p\right)} \prod_{i=1}^3 \frac{\zeta_p(1)}{\zeta_p(2)} \left(\frac{p^{k_i} \omega_{i,p}(p)}{\alpha_p(f_i)^2}\right)^n \\ &= Z_p^*(f_{\mathcal{D},s,p}) = \chi_p(-1) E_p\left(s + \frac{1}{2}, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \chi_p\right). \end{aligned}$$

Since  $\chi_{\mathbf{A}}$  is unramified outside  $p$ , we have  $\chi_\infty(-1) = \chi_p(-1)$ .  $\square$

**7.3. The congruence number.** Put  $\Delta = (\mathbf{Z}/Np\mathbf{Z})^\times$ . Let  $\hat{\Delta}$  be the group of Dirichlet characters modulo  $Np$ . Enlarging  $\mathcal{O}$  if necessary, we assume that every  $\chi \in \hat{\Delta}$  takes value in  $\mathcal{O}^\times$ . Let

$$\mathbf{S}^{\text{ord}}(N, \mathbf{I}) := \bigoplus_{\chi \in \hat{\Delta}} \mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I})$$

be the space of ordinary  $\mathbf{I}$ -adic cusp forms of tame level  $\Gamma_1(N)$ . Let  $\sigma_d$  denote the usual diamond operator for  $d \in \Delta$  acting on  $\mathbf{S}^{\text{ord}}(N, \mathbf{I})$  by  $\sigma_d(\mathbf{f})_{\chi \in \hat{\Delta}} = (\chi(d)\mathbf{f})_{\chi \in \hat{\Delta}}$ . The ordinary  $\mathbf{I}$ -adic cuspidal Hecke algebra  $\mathbf{T}(N, \mathbf{I})$  is defined as the  $\mathbf{I}$ -subalgebra of  $\text{End}_{\mathbf{I}} \mathbf{S}^{\text{ord}}(N, \mathbf{I})$  generated over  $\mathbf{I}$  by the Hecke operators  $T_\ell$  with  $\ell \nmid Np$ , the operators  $\mathbf{U}_\ell$  with  $\ell | Np$  and the diamond operators  $\sigma_d$  with  $d \in \hat{\Delta}$ . Let  $\mathbf{T}_k^{\text{ord}}(N, \chi)$  denote the  $\mathcal{O}$ -subalgebra of  $\text{End}_{\mathcal{O}} e_{\text{ord}} \mathcal{S}_k(N, \chi)$  generated over  $\mathcal{O}$  by the operators  $T_\ell$  with  $\ell \nmid Np$  and  $\mathbf{U}_\ell$  with  $\ell | Np$ .

Let  $\mathbf{f} \in \mathbf{S}^{\text{ord}}(N, \chi, \mathbf{I})$  be a primitive Hida family of tame conductor  $N$  and character  $\chi$ . The corresponding homomorphism  $\lambda_{\mathbf{f}} : \mathbf{T}(N, \mathbf{I}) \rightarrow \mathbf{I}$  is defined by  $\lambda_{\mathbf{f}}(T_\ell) = \mathbf{a}(\ell, \mathbf{f})$  for  $\ell \nmid Np$ ,  $\lambda_{\mathbf{f}}(\mathbf{U}_\ell) = \mathbf{a}(\ell, \mathbf{f})$  for  $\ell | Np$  and  $\lambda_{\mathbf{f}}(\sigma_d) = \chi(d)$  for  $d \in \Delta$ . We denote by  $\mathbf{m}_{\mathbf{f}}$  the maximal of  $\mathbf{T}(N, \mathbf{I})$  containing  $\text{Ker } \lambda_{\mathbf{f}}$  and by  $\mathbf{T}_{\mathbf{m}_{\mathbf{f}}}$  the localization of  $\mathbf{T}(N, \mathbf{I})$  at  $\mathbf{m}_{\mathbf{f}}$ . It is the local ring of  $\mathbf{T}(N, \mathbf{I})$  through which  $\lambda_{\mathbf{f}}$  factors. Recall that the congruence ideal  $C(\mathbf{f})$  of the morphism  $\lambda_{\mathbf{f}} : \mathbf{T}_{\mathbf{m}_{\mathbf{f}}} \rightarrow \mathbf{I}$  is defined by

$$C(\mathbf{f}) := \lambda_{\mathbf{f}}(\text{Ann}_{\mathbf{T}_{\mathbf{m}_{\mathbf{f}}}}(\text{Ker } \lambda_{\mathbf{f}})) \subset \mathbf{I}.$$

It is well-known that  $\mathbf{T}_{\mathbf{m}_f}$  is a local finite flat  $\Lambda$ -algebra, and there is an algebra direct sum decomposition

$$(7.1) \quad \tilde{\lambda}_f : \mathbf{T}_{\mathbf{m}_f} \otimes_{\mathbf{I}} \text{Frac } \mathbf{I} \simeq \text{Frac } \mathbf{I} \oplus \mathcal{B}, \quad t \mapsto \tilde{\lambda}_f(t) = (\lambda_f(t), \lambda_{\mathcal{B}}(t)),$$

where  $\mathcal{B}$  is some finite dimensional  $(\text{Frac } \mathbf{I})$ -algebra ([Hid88b, Corollary 3.7]). Then we have

$$C(\mathbf{f}) = \lambda_f(\mathbf{T}_{\mathbf{m}_f} \cap \tilde{\lambda}_f^{-1}(\text{Frac } \mathbf{I} \oplus \{0\}))$$

by definition. Now we impose the following hypothesis:

**Hypothesis (CR).** The residual Galois representation  $\bar{\rho}_f$  of  $\rho_f$  is absolutely irreducible and  $p$ -distinguished.

Under the hypothesis above  $\mathbf{T}_{\mathbf{m}_f}$  is Gorenstein by [Wil95, Corollary 2, page 482]. With this property of  $\mathbf{T}_{\mathbf{m}_f}$  Hida in [Hid88a] proved that the congruence ideal  $C(\mathbf{f})$  is generated by a non-zero element  $\eta_f \in \mathbf{I}$ , called the congruence number for  $\mathbf{f}$ . Let  $1_f^*$  be the unique element in  $\mathbf{T}_{\mathbf{m}_f} \cap \tilde{\lambda}_f^{-1}(\text{Frac } \mathbf{I} \oplus \{0\})$  such that  $\lambda_f(1_f^*) = \eta_f$ . Then  $1_f := \eta_f^{-1} 1_f^*$  is the idempotent in  $\mathbf{T}_{\mathbf{m}_f} \otimes_{\mathbf{I}} \text{Frac } \mathbf{I}$  corresponding to the direct summand  $\text{Frac } \mathbf{I}$  of (7.1) and  $1_f$  does not depend on any choice of a generator of  $C(\mathbf{f})$ . For  $d \in \widehat{\Delta}$  we write  $\wp_{Q, \chi}$  for the ideal of  $\mathbf{T}(N, \mathbf{I})$  generated by  $\wp_Q = \text{Ker } Q$  and  $\{\sigma_d - \chi(d)\}_{d \in \Delta}$ . A classical result in Hida theory asserts that

$$\mathbf{T}(N, \mathbf{I})/\wp_{Q, \chi} \simeq \mathbf{T}_{k_Q}^{\text{ord}}(Np^e, \chi \omega^{-k_Q} \epsilon_Q) \otimes_{\mathcal{O}} \mathcal{O}(Q)$$

(see Theorem 3.4 of [Hid88b]). Moreover, for each arithmetic point  $Q$ , it is also shown by Hida that the specialization  $\eta_f(Q) \in \mathcal{O}(Q)$  is the congruence number for  $\mathbf{f}_Q$  and

$$1_f := \eta_f^{-1} 1_f^* \pmod{\wp_{\chi, Q}} \in \mathbf{T}_{k_Q}^{\text{ord}}(Np^e, \chi \omega^{-k_Q} \epsilon_Q) \otimes_{\mathcal{O}} \text{Frac } \mathcal{O}(Q)$$

is the idempotent with  $\lambda_f(1_f) = 1$ .

**Definition 7.2.** Let  $\mathbf{f}$  be a primitive Hida family satisfying (CR). To each choice of the congruence number  $\eta_f$  we associate Hida's canonical period  $\Omega_f$  of a  $p$ -ordinary newform  $f$  of weight  $k$  obtained by the specialization of  $\mathbf{f}$  defined by

$$\Omega_f := \eta_f^{-1} \cdot (-2\sqrt{-1})^{k+1} \|f^\circ\|_{\Gamma_0(N_f^\circ)}^2 \cdot \mathcal{E}_p(f, \text{Ad}),$$

where  $\eta_f$  is the specialization of  $\eta_f$ ,  $f^\circ$  the primitive form associated with  $f$ ,  $N_f^\circ$  its conductor and  $\mathcal{E}_p(f, \text{Ad})$  the modified  $p$ -Euler factor attached to the adjoint motive of  $f$  (cf. [Hsi19, (3.10)]).

**7.4. Hida's functional.** When  $\varphi \in \mathcal{A}_k(N, \omega_{\mathbf{A}})$  and  $\varphi' \in \mathcal{A}_k(N, \omega_{\mathbf{A}}^{-1})$  are cuspidal, we define the pairing by

$$\langle \rho(\mathcal{J}_\infty)\varphi, \varphi' \rangle = \int_{\mathbf{A}^\times \text{GL}_2(\mathbf{Q}) \backslash \text{GL}_2(\mathbf{A})} \varphi(g\mathcal{J}_\infty)\varphi'(g) dg.$$

Let  $\chi$  be a Dirichlet character and let  $f \in \mathcal{S}_k(N_f, \chi)$  be an ordinary  $p$ -stabilized newform of level  $N_f$ , i.e.,  $\mathbf{U}_p f = \alpha_p(f)f$  with  $p$ -unit  $\iota_p^{-1}(\alpha_p(f))$ . Write  $N_f = N_t p^c$  with  $N_t$  prime to  $p$ . For  $n \geq c$ , we define Hida's functional  $L_f$  on  $\mathcal{S}_k(N_t p^{2n}, \chi; \mathcal{O})$  by

$$L_f(\mathcal{F}) = \left( \frac{\omega_{f,p}(p)p^k}{\alpha_p(f)^2} \right)^{n-1} \frac{\langle \rho(\mathcal{J}_\infty t_n)(\varphi \otimes \omega_{\mathbf{A}}^{-1}), \Phi(\mathcal{F}) \rangle}{\langle \rho(\mathcal{J}_\infty t_1)(\varphi \otimes \omega_{\mathbf{A}}^{-1}), \varphi \rangle},$$

where  $\omega_{\mathbf{A}}$  denotes the central character of the adèlic lift  $\varphi = \Phi(f)$  of  $f$ . Note that for  $\mathcal{F} \in \mathcal{S}_k(Np^{2n}, \chi)$  with  $N_t \mid N$ ,

$$L_f(\mathcal{F}) = [\Gamma_0(N) : \Gamma_0(N_t)]^{-1} L_f(\text{Tr}_{N/N_t} \mathcal{F}).$$

**Lemma 7.3.** (1)  $L_f(f) = 1$ .

(2) If  $\mathcal{F}_0 \in \mathcal{N}_{k+2m}(Np^{2n}, \chi)$  with  $N_t \mid N$ , then

$$\frac{L_f(1_f^* \text{Tr}_{N/N_t} e_{\text{ord}} \text{Hol}(\lambda_z^m \mathcal{F}_0))}{\zeta_{\mathbf{Q}}(2)[\text{SL}_2(\mathbf{Z}) : \Gamma_0(N)]} = (-1)^{m+1} (2\sqrt{-1})^{k+1} \frac{\langle \rho(\mathcal{J}_\infty t_n)(V_+^m \varphi \otimes \omega_{\mathbf{A}}^{-1}), \Phi(\mathcal{F}_0) \rangle}{\Omega_f \left( \frac{\alpha_p(f)^2}{p^k \omega_{f,p}(p)} \right)^n \frac{\zeta_p(2)}{\zeta_p(1)}}.$$

PROOF. The first assertion follows from the following formula stated in [Hsi19, Lemma 3.6]:

$$\begin{aligned} \langle \rho(\mathcal{J}_\infty t_n)(\varphi \otimes \omega_{\mathbf{A}}^{-1}), \varphi \rangle &= \omega_{f,\infty}(-1) \langle \rho(\mathcal{J}_\infty t_n) \varphi \otimes \omega_{\mathbf{A}}^{-1}, \varphi \rangle \\ &= \frac{(-1)^k \zeta_{\mathbf{Q}}(2)^{-1}}{[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N_t)]} \cdot \|f^\circ\|_{\Gamma_0(N_{f^\circ})}^2 \cdot \mathcal{E}_p(f, \mathrm{Ad}) \cdot \frac{\alpha_p(f)^{2n} \zeta_p(2)}{p^{kn} \omega_{f,p}(p)^n \zeta_p(1)} \\ &= -\frac{\zeta_{\mathbf{Q}}(2)^{-1}}{[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N_t)]} \cdot \frac{\eta_f \Omega_f}{(2\sqrt{-1})^{k+1}} \cdot \left( \frac{\alpha_p(f)^2}{p^k \omega_{f,p}(p)} \right)^n \frac{\zeta_p(2)}{\zeta_p(1)}. \end{aligned}$$

We remark that  $\check{\varphi} = \varphi$  and  $\omega_{(p)} = \omega_{\mathbf{A}}$  in the notation of [Hsi19].

To see the second part, we note that as a consequence of strong multiplicity one theorem for elliptic modular forms, the idempotent  $1_f = \eta_f^{-1} 1_f^*$  is generated by the Hecke operators  $T_\ell$  with  $\ell \nmid Np$ , which implies that  $1_f$  is the adjoint operator of  $1_{\varphi \otimes \omega_{\mathbf{A}}^{-1}}$  with respect to the pairing. We are thus led to  $L_f(\mathbf{1}_f^* \mathcal{F}) = \eta_f L_f(\mathcal{F})$ . Moreover,  $L_f(\mathbf{U}_p \mathcal{F}) = \alpha_p(f) L_f(\mathcal{F})$  (cf. the proof of Proposition 2.10 of [Kob13]) and hence

$$L_f(e_{\mathrm{ord}} \mathcal{F}) = \lim_{j \rightarrow \infty} L_f(\mathbf{U}_p^{j1} \mathcal{F}) = \lim_{j \rightarrow \infty} \alpha_p(f)^{j1} L_f(\mathcal{F}) = L_f(\mathcal{F}).$$

One can easily verify that for  $\phi \in \Phi(\mathcal{S}_k(M, \chi^{-1}))$ ,  $\mathcal{F}_1 \in \mathcal{N}_k(M, \chi)$  and  $\mathcal{F}_2 \in \mathcal{N}_{k+2}(M, \chi)$

$$\langle \rho(\mathcal{J}_\infty) \phi, \Phi(\mathrm{Hol} \mathcal{F}_1) \rangle = \langle \rho(\mathcal{J}_\infty) \phi, \Phi(\mathcal{F}_1) \rangle, \quad \langle \rho(\mathcal{J}_\infty) \phi, \Phi(\lambda_z \mathcal{F}_2) \rangle = -\langle \rho(\mathcal{J}_\infty) V_+ \phi, \Phi(\mathcal{F}_2) \rangle.$$

The second part is a consequence of these results.  $\square$

### 7.5. The construction of $p$ -adic triple product $L$ -functions. Let

$$\mathbf{F} = (\mathbf{f}, \mathbf{g}, \mathbf{h}) \in \mathbf{S}^{\mathrm{ord}}(N_1, \chi_1, \mathbf{I}) \times \mathbf{S}^{\mathrm{ord}}(N_2, \chi_2, \mathbf{I}) \times \mathbf{S}^{\mathrm{ord}}(N_3, \chi_3, \mathbf{I})$$

be a triplet of primitive  $\mathbf{I}$ -adic Hida families of tame square-free level  $(N_1, N_2, N_3)$  and tame characters  $(\chi_1, \chi_2, \chi_3)$ , where  $\mathbf{I}$  is a finite flat domain over  $\Lambda = \mathcal{O}[\Gamma]$ . Assuming that all  $\mathbf{f}$ ,  $\mathbf{g}$  and  $\mathbf{h}$  satisfy Hypothesis (CR), we fix a choice of the congruence numbers  $(\eta_f, \eta_g, \eta_h)$ . Let

$$\mathbf{1}_f^* \in \mathbf{T}(N_1, \mathbf{I}), \quad \mathbf{1}_g^* \in \mathbf{T}(N_2, \mathbf{I}), \quad \mathbf{1}_h^* \in \mathbf{T}(N_3, \mathbf{I})$$

be the idempotents multiplied by a fixed choice of congruence numbers  $(\eta_f, \eta_g, \eta_h)$  in the Hecke algebras attached to the newforms  $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ . Put

$$N := \mathrm{lcm}(N_1, N_2, N_3), \quad N^- := \mathrm{gcd}(N_1, N_2, N_3), \quad \mathbf{I}_3 := \mathbf{I} \widehat{\otimes}_{\mathcal{O}} \mathbf{I} \widehat{\otimes}_{\mathcal{O}} \mathbf{I}.$$

**Definition 7.4.** Define the  $p$ -adic triple product  $L$ -function  $L_{\mathbf{F},(a)}$  in  $\mathbf{I}_3[[T]]$  by

$$L_{\mathbf{F},(a)} := \text{the first Fourier coefficient of } \mathbf{1}_f^* \otimes \mathbf{1}_g^* \otimes \mathbf{1}_h^* (\mathrm{Tr}_{N/N_1} \otimes \mathrm{Tr}_{N/N_2} \otimes \mathrm{Tr}_{N/N_3} (\mathcal{G}_{\underline{\chi}}^{(a)})) \in \mathbf{I}_3[[T]].$$

We denote by  $V_{\mathbf{f}}$  the associated  $p$ -adic Galois representation, and by  $\mathrm{WD}_\ell(V_{\mathbf{f}_Q})$  the representation of the Weil-Deligne group  $W_{\mathbf{Q}_\ell}$  attached to  $V_{\mathbf{f}_Q}$  for each prime  $\ell$ . The epsilon factor of  $\mathbf{V}_{(\underline{Q}, P)}$  at  $\ell$  is defined by

$$\varepsilon_\ell(\mathbf{V}_{(\underline{Q}, P)}, s) = \varepsilon(s + k_P - w_{\underline{Q}}/2, \mathrm{WD}_\ell(V_{\mathbf{f}_{Q_1}}) \otimes \mathrm{WD}_\ell(V_{\mathbf{g}_{Q_2}}) \otimes \mathrm{WD}_\ell(V_{\mathbf{h}_{Q_3}}) \otimes \omega^{a-k_P} \epsilon_P, \psi_\ell).$$

By the assumption (sf) and the rigidity of automorphic types of Hida families  $\mathrm{WD}_\ell(V_{\mathbf{f}_{Q_1}})$ ,  $\mathrm{WD}_\ell(V_{\mathbf{g}_{Q_2}})$ ,  $\mathrm{WD}_\ell(V_{\mathbf{h}_{Q_3}})$  are either unramified or the Steinberg representation twisted by an unramified character. Moreover, for  $\ell \mid N_1$ , there is an unramified finite order character  $\xi_{\mathbf{f}, \ell} : G_{\mathbf{Q}_\ell} \rightarrow \overline{\mathbf{Q}}^\times$  such that  $\xi_{\mathbf{f}, \ell}^2 = \chi_{1, \ell}^{-1}$  and

$$V_{\mathbf{f}}|_{G_{\mathbf{Q}_\ell}} \simeq \begin{pmatrix} \xi_{\mathbf{f}, \ell} \varepsilon_{\mathrm{cyc}} \langle \varepsilon_{\mathrm{cyc}} \rangle_I^{-1/2} & * \\ 0 & \xi_{\mathbf{f}, \ell} \langle \varepsilon_{\mathrm{cyc}} \rangle_I^{-1/2} \end{pmatrix}.$$

Define  $\varepsilon_\ell(\mathbf{f})_{X_1} \in \mathbf{I}[[X_1]]$  by  $\varepsilon_\ell(\mathbf{f})_{X_1} = \xi_{\mathbf{f}, \ell}(\ell)^{-1} \langle \ell \rangle_{X_1}^{1/2}$ . Then

$$\varepsilon_\ell((1 - k_{Q_1})/2, \mathrm{WD}_\ell(V_{\mathbf{f}_{Q_1}}), \psi_\ell) = \varepsilon_\ell(\mathbf{f}_{Q_1}).$$

Let  $N_{\mathbf{V}} = N^- N^4$  be the tame level of  $\mathbf{V}$ , which independent of the choice of arithmetic specializations by the rigidity. We define the  $\mathbf{I}_4$ -adic root number  $\varepsilon^{(p^\infty)}(\mathbf{V}) \in \mathbf{I}_4^\times$  by

$$(7.2) \quad \varepsilon^{(p^\infty)}(\mathbf{V}) = \prod_{i=1}^3 \langle N_{\mathbf{V}} \rangle_{X_i} \cdot \omega(N_{\mathbf{V}})^{-a} N_{\mathbf{V}}^{-1} \langle N_{\mathbf{V}} \rangle_T^{-1} (\chi_1 \chi_2 \chi_3)(N^2) \prod_{\ell \mid N^-} \xi_{\mathbf{f}, \ell}(\ell)^{-1} \xi_{\mathbf{g}, \ell}(\ell)^{-1} \xi_{\mathbf{h}, \ell}(\ell)^{-1}.$$

**Lemma 7.5.** *Notation being as above, we get*

$$\varepsilon^{(p^\infty)}(\mathbf{V}_{(\underline{Q}, P)}) = \prod_{\ell \neq p} \varepsilon_\ell(\mathbf{V}_{(\underline{Q}, P)}, 0).$$

PROOF. We retain the notation of the proof of Proposition 6.8. Remark 3.5 gives

$$\varepsilon_\ell(\mathbf{V}_{(\underline{Q}, P)}, 0) = \chi_\ell(\ell)^4 \hat{\omega}(\ell)^2 \ell^{-4k_P + 2(k_{Q_1} + k_{Q_2} + k_{Q_3}) - 4} = \langle \ell^2 \rangle_{X_1, X_2, X_3, T}^{(a)}(\underline{Q}, P)$$

if  $\ell$  divides  $N/N^-$ . Put  $\xi_\ell = \xi_{\mathbf{f}, \ell} \xi_{\mathbf{g}, \ell} \xi_{\mathbf{h}, \ell}$ . If  $\ell$  divides  $N^-$ , then

$$\begin{aligned} \varepsilon_\ell(\mathbf{V}_{(\underline{Q}, P)}, 0) &= \chi_\ell(\ell)^5 \xi_\ell(\text{Frob}_\ell)^5 \ell^{5(-2k_P + k_{Q_1} + k_{Q_2} + k_{Q_3} - 2)/2} \\ &= \omega(\ell)^{-5a} (\chi_1 \chi_2 \chi_3)(\ell)^2 \xi_{\mathbf{f}, \ell}(\text{Frob}_\ell) (\ell)_T^{-5} \langle \ell \rangle_{X_1}^{5/2} \langle \ell \rangle_{X_2}^{5/2} \langle \ell \rangle_{X_3}^{5/2}(\underline{Q}, P) \ell^{-5}. \end{aligned}$$

We have thus completed our proof.  $\square$

**7.6. The interpolation formulae.** Let  $\mathbf{V} = V_{\mathbf{f}} \hat{\otimes}_{\mathcal{O}} V_{\mathbf{g}} \hat{\otimes}_{\mathcal{O}} V_{\mathbf{h}} \hat{\otimes}_{\mathcal{O}} \omega^a \langle \varepsilon_{\text{cyc}} \rangle_T$  be the triple tensor product of  $\mathbf{I}$ -adic Galois representations associated with primitive Hida families  $\mathbf{f}$ ,  $\mathbf{g}$  and  $\mathbf{h}$  twisted by  $\omega^a \langle \varepsilon_{\text{cyc}} \rangle_T$ . Define the rank four  $G_{\mathbf{Q}_p}$ -invariant subspace of  $\mathbf{V}$  by

$$\text{Fil}^+ \mathbf{V} = (\text{Fil}^0 V_{\mathbf{f}} \otimes \text{Fil}^0 V_{\mathbf{g}} \otimes V_{\mathbf{h}} + \text{Fil}^0 V_{\mathbf{f}} \otimes V_{\mathbf{g}} \otimes \text{Fil}^0 V_{\mathbf{h}} + V_{\mathbf{f}} \otimes \text{Fil}^0 V_{\mathbf{g}} \otimes \text{Fil}^0 V_{\mathbf{h}}) \otimes \omega^a \langle \varepsilon_{\text{cyc}} \rangle_T.$$

Recall that  $w_{\underline{Q}} = k_{Q_1} + k_{Q_2} + k_{Q_3} - 3$  and  $\Gamma_{\mathbf{V}_{(\underline{Q}, P)}}(s) = L_\infty \left( s + k_P - \frac{w_{\underline{Q}}}{2}, \pi_{\mathbf{f}_{Q_1}} \times \pi_{\mathbf{g}_{Q_2}} \times \pi_{\mathbf{h}_{Q_3}} \right)$ . The modified  $p$ -Euler factor  $\mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{(\underline{Q}, P)})$  is defined in the introduction.

**Theorem 7.6.** *Let  $p > 3$ . Assume that  $N := \text{lcm}(N_1, N_2, N_3)$  is square-free and that the conductor of tame nebentypus  $\chi_i$  divides  $p$ . Let  $t$  denote the number of prime factors of  $N$ . If  $\mathbf{f}$ ,  $\mathbf{g}$  and  $\mathbf{h}$  satisfy Hypothesis (CR), then for each arithmetic point  $(\underline{Q}, P) = (Q_1, Q_2, Q_3, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$  we have*

$$L_{\mathbf{F}, (a)}(\underline{Q}, P) = \Gamma_{\mathbf{V}_{(\underline{Q}, P)}}(0) \cdot \frac{L(\mathbf{V}_{(\underline{Q}, P)}, 0)}{\Omega_{\mathbf{f}_{Q_1}} \Omega_{\mathbf{g}_{Q_2}} \Omega_{\mathbf{h}_{Q_3}}} \cdot (\sqrt{-1})^{k_{Q_1} + k_{Q_2} + k_{Q_3} - 3} \cdot \mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{(\underline{Q}, P)}) \cdot \mathfrak{f}_{\underline{X}, a, N_1, N_2, N_3}(\underline{Q}, P),$$

where  $\mathfrak{f}_{\underline{X}, a, N_1, N_2, N_3} \in \mathbf{I}_4^\times$  is given by

$$\mathfrak{f}_{\underline{X}, a, N_1, N_2, N_3} := \frac{(-1)^t}{N} \prod_{\ell | N} (\langle \ell \rangle_{X_1, X_2, X_3, T}^{(a)})^2 \varepsilon_\ell(\mathbf{f} \otimes \mathbf{g} \otimes \mathbf{h} \otimes \omega^a \langle \varepsilon_{\text{cyc}} \rangle_T)^{-1}.$$

PROOF. For brevity we write  $(f_1, f_2, f_3) = (\mathbf{f}_{Q_1}, \mathbf{g}_{Q_2}, \mathbf{h}_{Q_3})$ ,  $(k, l, m) = (k_{Q_1}, k_{Q_2}, k_{Q_3})$ ,  $\pi_i = \pi_{f_i}$  and  $N_i = N_{f_i}$ . We may assume that  $k \geq l \geq m$ . Denote the parity type of  $(k, l, m)$  by  $\lambda$ . Put

$$\chi = \varepsilon_P \omega^{a-k_P}, \quad \omega_i = \omega^{k_{Q_i}} \chi_i^{-1} \varepsilon_{Q_i}^{-1}, \quad \mathcal{D} = (\chi, \omega_1^{-1}, \omega_2^{-1}, \omega_3^{-1}), \quad n = \max\{1, c(\omega_i), c(\chi)\}.$$

We define the functional  $L_{f_1, f_2, f_3}$  on

$$\mathcal{S}_k(N_1 p^{2n}, \omega_1^{-1}; \mathcal{O}(Q_1)) \otimes_{\mathcal{O}} \mathcal{S}_l(N_2 p^{2n}, \omega_2^{-1}; \mathcal{O}(Q_2)) \otimes_{\mathcal{O}} \mathcal{S}_m(N_3 p^{2n}, \omega_3^{-1}; \mathcal{O}(Q_3))$$

by

$$L_{f_1, f_2, f_3}(\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3) = L_{f_1}(\mathcal{F}_1) L_{f_2}(\mathcal{F}_2) L_{f_3}(\mathcal{F}_3).$$

Let  $1_{f_1}^*$  be the specialization of  $\mathbf{1}_{\mathbf{f}}^*$  at  $Q_1$ . By definition and the theory of newforms

$$1_{f_1}^* \otimes 1_{f_2}^* \otimes 1_{f_3}^* (\text{Tr}_{N/N_1} \otimes \text{Tr}_{N/N_2} \otimes \text{Tr}_{N/N_3} (\mathcal{G}_{\underline{X}}^{(a)}(\underline{Q}, P))) = L_{\mathbf{F}, (a)}(\underline{Q}, P) \cdot f_1 \otimes f_2 \otimes f_3.$$

We apply the functional  $L_{f_1, f_2, f_3}$  to both the sides to get

$$L_{\mathbf{F}, (a)}(\underline{Q}, P) = L_{f_1, f_2, f_3}(1_{f_1}^* \otimes 1_{f_2}^* \otimes 1_{f_3}^* (\text{Tr}_{N/N_1} \otimes \text{Tr}_{N/N_2} \otimes \text{Tr}_{N/N_3} (\mathcal{G}_{\underline{X}}^{(a)}(\underline{Q}, P))),$$

taking Lemma 7.3(1) into account. Let  $\varphi_i = \Phi(f_i)$  and  $G_{\mathbf{A}}(\mathcal{D}) = \Phi(\mathcal{G}_{\underline{X}}^{(a)}(\underline{Q}, P))$  be the adèlic lifts. Put  $\check{\varphi}_i = \varphi_i \otimes \omega_{i, \mathbf{A}}^{-1}$ . In the previous section we verified that

$$G_{\mathbf{A}}(\mathcal{D}) = \lim_{s \rightarrow -r + \frac{k-\lambda_1-1}{2}} (-1)^{k + \frac{l+m+\lambda_1}{2} + \lambda_2} e_{\text{ord}} \text{Hol} \left( \left( 1 \otimes V_-^{\frac{k-l-\lambda_2}{2}} \otimes V_-^{\frac{k-m-\lambda_3}{2}} \right) \iota^* E_{\mathbf{A}}^*(f_{\mathcal{D}, s, N}^{[k, \lambda]}), \right)$$

where  $r = k - k_P + \frac{l+m-\lambda_1}{2} - 2$  (see Proposition 6.8 and Definitions 6.2, 6.5). Lemma 7.3 (2) therefore gives

$$\begin{aligned} \frac{L_{\mathbf{F},(a)}(\underline{Q}, P)}{\zeta_{\mathbf{Q}}(2)^3 [\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)]^3} &= -(2\sqrt{-1})^{k+l+m+3} \frac{\zeta_p(1)^3}{\zeta_p(2)^3} \prod_{i=1}^3 \Omega_{f_i}^{-1} \left( \frac{p^{k_{Q_i}} \omega_{i,p}(p)}{\alpha_p(f_i)^2} \right)^n \\ &\times 4 \lim_{s \rightarrow k_P - \frac{k+l+m}{2} + 1} Z \left( \rho(\mathcal{J}_\infty t_n) \check{\varphi}_1, \rho(\mathcal{J}_\infty t_n) V_+^{\frac{k-l-\lambda_2}{2}} \check{\varphi}_2, \rho(\mathcal{J}_\infty t_n) V_+^{\frac{k-m-\lambda_3}{2}} \check{\varphi}_3, E_{\mathbf{A}}^*(f_{\mathcal{D},s,N}^{[k,\lambda]}) \right). \end{aligned}$$

Let  $W(\varphi_i) = \prod_v W_{i,v}$  be the Whittaker function of  $\varphi_i$ . Put  $\check{W}_{i,v} := W_{i,v} \otimes \omega_{i,v}^{-1}$ . Let  $\pi_i$  be the automorphic representation generated by  $\varphi_i$ . Writing  $N = \prod_{\ell|N} \ell$ , we finally get

$$L_{\mathbf{F},(a)}(\underline{Q}, P) = \frac{L\left(k_P - \frac{k+l+m-3}{2}, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}}\right) \mathcal{E}_p(\mathrm{Fil}^+ \mathbf{V} \otimes \epsilon_P \omega^{a-k_P}, k_P)}{(\sqrt{-1})^{3-(k+l+m)} \Omega_{f_1} \Omega_{f_2} \Omega_{f_3}} \prod_{\ell|N} Z_\ell^*$$

by Lemma 7.1, where

$$Z_\ell^* = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(\ell)]^3 \lim_{s \rightarrow k_P - \frac{k+l+m}{2} + 1} \frac{Z(\check{W}_{1,\ell}, \check{W}_{2,\ell}, \check{W}_{3,\ell}, f_{\mathcal{D},s,\ell})}{L\left(s + \frac{1}{2}, \pi_{1,\ell} \times \pi_{2,\ell} \times \pi_{3,\ell} \otimes \chi_\ell\right)}.$$

Proposition 3.4 gives

$$Z_\ell^* = -\ell(\hat{\omega}_\ell^2 \chi_\ell^4)(\ell) |\ell|^{4k_P - 2(k+l+m) + 4} \epsilon \left( k_P - \frac{k+l+m-3}{2}, \pi_{1,\ell} \times \pi_{2,\ell} \times \pi_{3,\ell} \otimes \chi, \psi_\ell \right)^{-1}.$$

By what we have seen in the proof of Proposition 6.8

$$\chi_\ell^2 \hat{\omega}_\ell(\ell) \ell^{-2k_P + (k+l+m) - 2} = \langle \ell \rangle_{X_1, X_2, X_3, T}(\underline{Q}, P).$$

This completes the proof.  $\square$

**Definition 7.7.** We normalize  $p$ -adic triple product  $L$ -function by

$$L_{\mathbf{F},(a)}^* := L_{\mathbf{F},(a)} \cdot \mathfrak{f}_{\chi,a,N_1,N_2,N_3}^{-1}.$$

**Remark 7.8.** Provided that  $p > 3$ ,  $\chi_1 \chi_2 \chi_3 = \omega^{2a}$  for some  $a$ , a three-variable  $p$ -adic  $L$ -function  $\mathcal{L}_{\mathbf{F}}^{\mathrm{bal}} \in \mathbf{I}_3$  was constructed by a different approach in [Hsi19, Theorem B] such that for each balanced central point  $\underline{Q} = (Q_1, Q_2, Q_3) \in \mathfrak{X}_{\mathbf{I}_3}^{\mathrm{bal}}$

$$\left(\mathcal{L}_{\mathbf{F}}^{\mathrm{bal}}(\underline{Q})\right)^2 = \Gamma_{\mathbf{V}_{\underline{Q}}}(\mathbf{0}) \cdot \frac{L(\mathbf{V}_{\underline{Q}}^\dagger, \mathbf{0})}{\Omega_{f_{Q_1}} \Omega_{g_{Q_2}} \Omega_{h_{Q_3}}} \cdot (\sqrt{-1})^{k_{Q_1} + k_{Q_2} + k_{Q_3} - 3} \cdot \mathcal{E}_p(\mathrm{Fil}^+ \mathbf{V}_{\underline{Q}}^\dagger),$$

where

$$\begin{aligned} \mathbf{V}^\dagger &:= \mathcal{V} \otimes \omega^a \langle \epsilon_{\mathrm{cyc}} \rangle_{\mathbf{X}_1}^{1/2} \langle \epsilon_{\mathrm{cyc}} \rangle_{\mathbf{X}_2}^{1/2} \langle \epsilon_{\mathrm{cyc}} \rangle_{\mathbf{X}_3}^{1/2} \epsilon_{\mathrm{cyc}}^{-1}, \\ \mathrm{Fil}^+ \mathbf{V}^\dagger &= \mathrm{Fil}^+ \mathcal{V} \otimes \omega^a \langle \epsilon_{\mathrm{cyc}} \rangle_{\mathbf{X}_1}^{1/2} \langle \epsilon_{\mathrm{cyc}} \rangle_{\mathbf{X}_2}^{1/2} \langle \epsilon_{\mathrm{cyc}} \rangle_{\mathbf{X}_3}^{1/2} \epsilon_{\mathrm{cyc}}^{-1}. \end{aligned}$$

We remark that  $\det V_f = (\chi_1 \circ \epsilon_{\mathrm{cyc}})^{-1} \langle \epsilon_{\mathrm{cyc}} \rangle_{\mathbf{I}}^{-1} \epsilon_{\mathrm{cyc}}$ . By the interpolation formulae, we find that

$$L_{\mathbf{F},(a-1)}^*(X_1, X_2, X_3, \mathbf{u}^{-1} \{(1+X_1)(1+X_2)(1+X_3)\}^{1/2} - 1) = \mathcal{L}_{\mathbf{F}}^{\mathrm{bal}}(X_1, X_2, X_3)^2.$$

This shows that the compatibility between  $p$ -adic  $L$ -functions constructed by different methods.

Without Hypothesis (CR) and the assumption  $p > 3$ , our method yields the construction of the  $p$ -adic  $L$ -function with denominators. For each  $p$ -stabilized newform  $f$  of weight  $k$ , define the modified period by

$$\Omega_f^b := (-2\sqrt{-1})^{k+1} \cdot \|f^\circ\|_{\Gamma_0(N_{f^\circ})}^2 \cdot \mathcal{E}_p(f, \mathrm{Ad}).$$

By definition,  $\Omega_f^b \cdot \eta_f$  is equal to Hida's canonical period  $\Omega_f$  up to  $p$ -adic units.

**Corollary 7.9.** *Let  $p > 2$ . There exists an element*

$$L_{\mathbf{F},(a)}^{**} \in \mathbf{I}_4 \otimes_{\mathbf{I}_3} (\mathrm{Frac} \mathbf{I} \otimes \mathrm{Frac} \mathbf{I} \otimes \mathrm{Frac} \mathbf{I})$$

such that

- for any  $H_1, H_2$  and  $H_3$  in the congruence ideals of  $\mathbf{f}, \mathbf{g}$  and  $\mathbf{h}$ ,

$$H_1 H_2 H_3 \cdot L_{\mathbf{F},(a)}^{**} \in \mathbf{I}_4;$$

- for each balanced critical  $(\underline{Q}, P) = (Q_1, Q_2, Q_3, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$ ,

$$L_{\mathbf{F},(a)}^{**}(\underline{Q}, P) = \frac{\Gamma_{\mathbf{V}(\underline{Q}, P)}(0) L(\mathbf{V}(\underline{Q}, P), 0)}{\Omega_{\mathbf{f}_{Q_1}}^b \Omega_{\mathbf{g}_{Q_2}}^b \Omega_{\mathbf{h}_{Q_3}}^b} \cdot (\sqrt{-1})^{k_{Q_1} + k_{Q_2} + k_{Q_3} - 3} \cdot \mathcal{E}_p(\text{Fil}^+ \mathbf{V}(\underline{Q}, P)),$$

PROOF. For any  $H_1, H_2$  and  $H_3$  in the congruence ideals of  $\mathbf{f}, \mathbf{g}$  and  $\mathbf{h}$ , we let  $L_H \in \mathbf{I}_3[[T]]$  be the first Fourier coefficient of

$$H_1 \mathbf{1}_{\mathbf{f}} \otimes H_2 \mathbf{1}_{\mathbf{g}} \otimes H_3 \mathbf{1}_{\mathbf{h}} \left( \text{Tr}_{N/N_1} \otimes \text{Tr}_{N/N_2} \otimes \text{Tr}_{N/N_3} (\mathcal{G}_{\underline{\chi}}^{(a)}) \right) \in \mathbf{I}_3[[T]][[q_1, q_2, q_3]].$$

Then  $L_{\mathbf{F},(a)}^{**} := L_H \cdot (H_1 H_2 H_3)^{-1} \cdot \mathfrak{f}_{\underline{\chi}, a, N_1, N_2, N_3}^{-1}$  enjoys the desired properties.  $\square$

This  $p$ -adic  $L$ -function  $L_{\mathbf{F},(a)}^{**}$  is more canonical in the sense that it does not depend on any particular choice of generators of the congruence ideal of  $\mathbf{f}, \mathbf{g}$  and  $\mathbf{h}$ .

**7.7. The functional equation.** Recall that we have fixed the topological generator  $\mathbf{u} = 1 + p$  of  $\Gamma = 1 + p\mathbf{Z}_p$  as in §6.5.

**Proposition 7.10.** *Assume that  $\chi_1 \chi_2 \chi_3 = \omega^{a_0}$ . Then*

$$L_{\mathbf{F},(a)}^*(X_1, X_2, X_3, T) = (-\varepsilon^{(p\infty)}(\mathbf{V})) \cdot L_{\mathbf{F},(a_0-a-2)}^* \left( X_1, X_2, X_3, \frac{(1+X_1)(1+X_2)(1+X_3)}{\mathbf{u}^2(1+T)} - 1 \right).$$

PROOF. Recall that  $\chi = \epsilon_P \omega^{a-k_P}$  and  $\omega_i = \chi_i^{-1} \epsilon_{Q_i}^{-1} \omega^{k_{Q_i}}$ . Put

$$k_{\check{P}} = k_{Q_1} + k_{Q_2} + k_{Q_3} - k_P - 2, \quad \epsilon_{\check{P}} = \epsilon_P^{-1} \epsilon_{Q_1} \epsilon_{Q_2} \epsilon_{Q_3}, \quad \check{\chi} = \epsilon_{\check{P}} \omega^{a_0-a-2-k_{\check{P}}} = \chi^{-1} \omega_1^{-1} \omega_2^{-1} \omega_3^{-1}.$$

Thus the left hand side specialized at  $(\underline{Q}, \check{P})$  equals

$$L_{\mathbf{F},(a_0-a-2)}^*(\underline{Q}, \check{P}) = \frac{L(1-s_0, \pi_1^{\vee} \times \pi_2^{\vee} \times \pi_3^{\vee} \otimes \chi_{\mathbf{A}}^{-1})}{(\sqrt{-1})^{3-(k_{Q_1}+k_{Q_2}+k_{Q_3})} \Omega_{f_1} \Omega_{f_2} \Omega_{f_3}} E_p(1-s_0, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \chi_p),$$

where  $s_0 = k_P - \frac{k_{Q_1} + k_{Q_2} + k_{Q_3} - 3}{2} = 1 - \left( k_{\check{P}} - \frac{k_{Q_1} + k_{Q_2} + k_{Q_3} - 3}{2} \right)$ .

Since  $(k_{Q_1}, k_{Q_2}, k_{Q_3})$  is balanced, we know that

$$\varepsilon(s, \pi_{1,\infty} \times \pi_{2,\infty} \times \pi_{3,\infty} \otimes \chi_{\infty}) = (-1)^{k_{Q_1} + k_{Q_2} + k_{Q_3} + 1} = -\hat{\omega}_{\infty}(-1) = -\hat{\omega}_p(-1).$$

By the global functional equation we get

$$L_{\mathbf{F},(a_0-a-2)}^*(\underline{Q}, \check{P}) = \frac{L(s_0, \pi_1 \times \pi_2 \times \pi_3 \otimes \chi_{\mathbf{A}})}{(\sqrt{-1})^{3-(k_{Q_1}+k_{Q_2}+k_{Q_3})} \Omega_{f_1} \Omega_{f_2} \Omega_{f_3}} \cdot \frac{-E_p(s_0, \pi_{1,p} \times \pi_{2,p} \times \pi_{3,p} \otimes \check{\chi}_p)}{\prod_{\ell \neq p} \varepsilon(s_0, \pi_{1,\ell} \times \pi_{2,\ell} \times \pi_{3,\ell} \otimes \chi_{\ell}, \psi_{\ell})}$$

in view of Lemma 2.7.  $\square$

## 8. THE TRIVIAL ZERO FOR THE TRIPLE PRODUCT OF ELLIPTIC CURVES

**8.1. The cyclotomic  $p$ -adic triple product  $L$ -functions for elliptic curves.** Let  $\mathbf{E} = E_1 \times E_2 \times E_3$  be the triple fiber product of rational elliptic curves  $E_i$  of square-free conductor  $M_i$  for  $i = 1, 2, 3$ . We denote the prime  $p$ -part of  $M_i$  by  $N_i$ . Recall the rank eight  $p$ -adic Galois representation  $\mathbf{V}_{\mathbf{E}}$  defined in (1.1). We write  $L(\mathbf{E} \otimes \chi, s)$  for the complex  $L$ -series attached to  $\mathbf{V}_{\mathbf{E}}$  twisted by a Dirichlet character  $\chi$ . Let  $M$  (resp.  $N$ ) and  $M^-$  (resp.  $N^-$ ) be the least common multiple and the greatest common divisor of  $M_1, M_2, M_3$  (resp.  $N_1, N_2, N_3$ ).

**Remark 8.1.** Let  $\Sigma^-$  be the set of prime factors  $\ell$  of  $M^-$  such that  $a_{\ell}(E_1) a_{\ell}(E_2) a_{\ell}(E_3) = 1$ . From Remark 3.5,  $\varepsilon(\mathbf{E}) = -(-1)^{\#\Sigma^-}$  is the sign in the functional equation for  $L(s, \mathbf{E})$ . From the formula (7.2) for the  $p$ -adic root number the  $p$ -adic sign  $\varepsilon_p(\mathbf{E}) = -\varepsilon^{(p\infty)}(\mathbf{V}_{\mathbf{E}}(2))$  differs from  $\varepsilon(\mathbf{E})$  if and only if  $p \in \Sigma^-$ .

Let  $f_i^\circ = \sum_{n=1}^{\infty} a_n(E_i)q^n \in \mathcal{S}_2(M_i, 1; \mathbf{Z})$  be the primitive Hecke eigenform associated with the  $p$ -adic Galois representation  $\mathbb{H}_{\text{ét}}^1(E_i/\overline{\mathbf{Q}}, \mathbf{Q}_p)$  by Wiles' modularity theorem. Hereafter, we assume that  $E_i$  has either good ordinary reduction or multiplicative reduction at  $p$ . Let  $f_i \in \mathcal{S}_2(pM_i, 1; \mathbf{Z}_p)$  be the  $p$ -stabilization of  $f_i^\circ$  (see (5.1)). If  $p$  and  $M_i$  are coprime, then  $\alpha_i = \alpha_p(f_i) \in \mathbf{Z}_p^\times$  denotes the  $p$ -adic unit root of the Hecke polynomial  $X^2 - a_p(E_i)X + p$  while if  $p$  divides  $M_i$ , then  $\alpha_i = a_p(E_i)$ . Define a period and a fudge factor by

$$\Omega(\mathbf{E}) = \prod_{i=1}^3 \Lambda(1, E_i, \text{Ad}), \quad c_p = \prod_{i=1}^3 \mathcal{E}_p(f_i, \text{Ad}),$$

where  $\Lambda(s, E_i, \text{Ad})$  denotes the complete adjoint  $L$ -function for  $f_i$

Let  $\mathbf{T}_i = \mathbf{T}(N_i, \Lambda)$  be the big cuspidal ordinary Hecke algebra over  $\Lambda = \mathbf{Z}_p[[X]]$  with  $X = [\mathbf{u}] - 1$ . Each  $f_i$  induces a surjective homomorphism  $\lambda_{f_i} : \mathbf{T}_i \rightarrow \mathbf{Z}_p$ . Let  $\mathfrak{m}_i$  be the maximal ideal of  $\mathbf{T}_i$  containing  $\ker \lambda_{f_i}$  and  $\mathbf{I}_i = (\mathbf{T}_i)_{\mathfrak{m}_i}$  be the localization at  $\mathfrak{m}_i$ . Let  $\mathbf{f}_i = \sum_{n=1}^{\infty} \mathbf{a}(n, \mathbf{f}_i)q^n \in \mathbf{S}(N_i, \omega^2, \mathbf{I}_i)$  be the primitive Hida family of tame level  $N_i$  such that  $f_i$  is the specialization  $\mathbf{f}_{i, Q_i^\circ}$  at some arithmetic point  $Q_i^\circ$  with  $k_{Q_i^\circ} = 2$  and  $\epsilon_{Q_i^\circ} = 1$ . Now we consider the four-variable  $p$ -adic  $L$ -function  $L_{\mathbf{F}, (2)}^{**}$  in Corollary 7.9 with  $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$  and  $a = 2$ . Define the cyclotomic  $p$ -adic  $L$ -function by

$$L_p(\mathbf{E}, T) := c_p \cdot L_{\mathbf{F}, (2)}^{**}(Q_1^\circ, Q_2^\circ, Q_3^\circ, \mathbf{u}^2(1+T) - 1) \in \mathbf{Z}_p[[\text{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]] \otimes \mathbf{Q}_p.$$

**Proposition 8.2.** *The element  $L_p(\mathbf{E}) \in \mathbf{Z}_p[[\text{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]] \otimes \mathbf{Q}_p$  satisfies the following interpolation property*

$$\hat{\chi}(L_p(\mathbf{E})) = \frac{L(\mathbf{E} \otimes \hat{\chi}, 2)}{2^4 \pi^5 \Omega(\mathbf{E})} \mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{\mathbf{E}} \otimes \hat{\chi})$$

for all finite-order characters  $\hat{\chi}$  of  $\text{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ . Moreover, it satisfies the functional equation

$$L_p(\mathbf{E}, T) = \varepsilon_p(\mathbf{E}) \langle N^- N^4 \rangle_T^{-1} L_p(\mathbf{E}, (1+T)^{-1} - 1).$$

PROOF. Define  $(Q^\circ, P) = (Q_1^\circ, Q_2^\circ, Q_3^\circ, P) \in \mathfrak{X}_{\mathbf{I}_4}^{\text{bal}}$  with  $Q_i^\circ$  as above,  $k_P = 2$  and  $\epsilon_P = \hat{\chi}$ . Then  $\mathbf{V}_{(Q^\circ, P)} = \mathbf{V}_{\mathbf{E}}(2) \otimes \hat{\chi}$  and  $\hat{\chi}(L_p(\mathbf{E})) = c_p \cdot L_{\mathbf{F}, (2)}^{**}(Q^\circ, P)$ . The assertions follows from Corollary 7.9, Proposition 7.10 and the equation  $2^2 \|f_i^\circ\| = \Lambda(1, E_i, \text{Ad})$  ([Hsi19, (2.18)]).  $\square$

**8.2. The trivial zero conjecture for the triple product of elliptic curves.** We prove the trivial zero conjecture for the cyclotomic  $p$ -adic triple product  $L$ -function. We define a function on  $\mathbf{Z}_p$  by

$$L_p(\mathbf{E}, s) := L_p(\mathbf{E}, \mathbf{u}^s - 1).$$

We consider the case where  $L_p(\mathbf{E}, s)$  has a trivial zero at the critical value  $s = 2$ . By Remark 8.3 below we essentially only need to consider the following two cases:

- (i) all  $E_1, E_2$  and  $E_3$  have multiplicative reduction at  $p$  such that  $\alpha_1 \alpha_2 \alpha_3 = 1$ .
- (ii)  $E_1$  has multiplicative reduction at  $p$ ;  $E_2$  and  $E_3$  have good ordinary reduction at  $p$  such that  $\alpha_2 = \alpha_1 \alpha_3$ .

**Remark 8.3.** Let  $\beta_i = p\alpha_i^{-1}$ . Then  $\mathcal{E}_p(\text{Fil}^+ \mathbf{V}_{\mathbf{E}}(2)) = 0$  if and only if  $L_p((\text{Fil}^+ \mathbf{V}_{\mathbf{E}}(2))^\vee, 1)^{-1} = 0$  if and only if one of the following equations holds:

$$\beta_1 \beta_2 \beta_3 = p^2, \quad \beta_1 \beta_2 \alpha_3 = p^2, \quad \beta_1 \alpha_2 \beta_3 = p^2, \quad \alpha_1 \beta_2 \beta_3 = p^2.$$

The ordinality hypothesis rules out the first equation. The Ramanujan conjecture forces one or all of  $E_i$  to have multiplicative reduction at  $p$ . When  $E_1$  is multiplicative at  $p$ , we will have  $\alpha_1 \in \{\pm 1\}$  and  $\alpha_2 = \alpha_1 \alpha_3$ .

In the above cases (i) and (ii), the trivial zero conjecture predicts that the leading coefficient of the Taylor expansion of  $L_p(\mathbf{E}, s)$  at  $s = 2$  should be essentially the product of Greenberg's  $\mathcal{L}$ -invariant for  $\mathbf{E}$  and the central value  $L(\mathbf{E}, 2)$ . Note that the localization of  $\mathbf{I}_i$  at  $Q_i^\circ$  is that of  $\Lambda$  at  $P_2$ , where  $P_2$  is the principal ideal generated by  $(1+X)\mathbf{u}^{-2} - 1$ , so  $\mathbf{I}_i$  is contained in  $\Lambda[\frac{1}{t_i}]$  with some  $t_i(\mathbf{u}^2 - 1) \neq 0$ . In what follows, we shall replace  $\mathbf{I}_i$  by  $\Lambda[t_i^{-1}]$  with some  $t_i(\mathbf{u}^2 - 1) \neq 0$ . Let  $\mathcal{U} \subset \mathbf{Z}_p$  be a neighborhood around 0 such that  $(t_1 t_2 t_3)(\mathbf{u}^{s+2} - 1) \neq 0$  for any  $s \in \mathcal{U}$ . To introduce Greenberg's  $\mathcal{L}$ -invariants, we let

$$\mathbf{a}_i(s) := \mathbf{a}(p, \mathbf{f}_i)(\mathbf{u}^{s+2} - 1); \quad \ell_i := \alpha_i^{-1} \cdot \left. \frac{d\mathbf{a}_i(s)}{ds} \right|_{s=0} \quad (s \in \mathcal{U}).$$



Note that  $\mathbf{a}_i(0) = \alpha_i$  by definition. If  $\alpha_i = 1$ , then  $-2\ell_i = \frac{\log_p q_{E_i}}{\text{ord}_p q_{E_i}}$  by [GS93, Theorem 3.18]. According to the discussion in [Gre94b, §3], Greenberg's  $\mathcal{L}$ -invariant for the Galois representation (1.1) is given by

$$\mathcal{L}_p(\mathbf{E}) := \begin{cases} -8\ell_1\ell_2\ell_3 & \text{in Case (i);} \\ 4\ell_1^2 & \text{in Case (ii).} \end{cases}$$

The non-vanishing of these  $\mathcal{L}$ -invariants is known, thanks to the work [BSDGP96]. The aim of this section is to prove the following:

**Theorem 8.4** (Trivial zero conjecture). *(1) In Case (i),  $\text{ord}_{s=2}L_p(\mathbf{E}, s) \geq 3$ , and*

$$\left. \frac{L_p(\mathbf{E}, s)}{(s-2)^3} \right|_{s=2} = \mathcal{L}_p(\mathbf{E}) \cdot \frac{L(\mathbf{E}, 2)}{2^4\pi^5\Omega(\mathbf{E})}.$$

*(2) In Case (ii),  $\text{ord}_{s=2}L_p(\mathbf{E}, s) \geq 2$  and*

$$\left. \frac{L_p(\mathbf{E}, s)}{(s-2)^2} \right|_{s=2} = \mathcal{L}_p(\mathbf{E})(-p\alpha_2^{-2})(1-\alpha_2^{-2})^2 \cdot \frac{L(\mathbf{E}, 2)}{2^4\pi^5\Omega(\mathbf{E})}.$$

**8.3. Improved  $p$ -adic  $L$ -functions.** We define an analytic function on  $\mathcal{U}^3 \times \mathbf{Z}_p \subset \mathbf{Z}_p^4$  by

$$L_p(x, y, z, s) := c_p \cdot \langle N^- N^4 \rangle^{\frac{2s-(x+y+z)}{4}} L_{\mathbf{F},(2)}^{**}(\mathbf{u}^{x+2} - 1, \mathbf{u}^{y+2} - 1, \mathbf{u}^{z+2} - 1, \mathbf{u}^{s+2} - 1),$$

which satisfies

$$(8.1) \quad L_p(0, 0, 0, s) = \langle N^- N^4 \rangle^{s/2} L_p(\mathbf{E}, s+2), \quad L_p(x, y, z, s) = \varepsilon_p(\mathbf{E}) \cdot L_p(x, y, z, x+y+z-s).$$

To follow the method used in [GS93] (*cf.* [BDJ17]), we introduce  $p$ -adic  $L$ -functions which have only less variables but have better interpolation properties.

**Lemma 8.5** (Improved  $p$ -adic  $L$ -functions). *Suppose that  $f_1^\circ$  is special at  $p$ , i.e.  $\alpha_1 = \mathbf{a}_1(0) = \pm 1$ .*

*(1) There exist a two-variable improved  $p$ -adic  $L$ -function  $L_p^\dagger(x, s)$  and a one-variable improved  $p$ -adic  $L$ -function  $L_p^{\dagger\dagger}(s)$  such that*

$$L_p(x, s, s, s) = \left(1 - \frac{\mathbf{a}_2(s)}{\mathbf{a}_1(x)\mathbf{a}_3(s)}\right) \left(1 - \frac{\mathbf{a}_3(s)}{\mathbf{a}_1(x)\mathbf{a}_2(s)}\right) L_p^\dagger(x, s), \quad L_p^\dagger(s, s) = \left(1 - \frac{\mathbf{a}_1(s)}{\mathbf{a}_2(s)\mathbf{a}_3(s)}\right) L_p^{\dagger\dagger}(s).$$

*(2) For any positive integer  $k$  with  $k \equiv 2 \pmod{p-1}$  and  $k-2 \in \mathcal{U}$ , we have the interpolation formula*

$$L_p^\dagger(0, k-2) = \mathcal{E}^\dagger(k-2) \cdot \frac{\Gamma(k-1)\Gamma(k)}{2^{2k-3}(\pi\sqrt{-1})^{2k+1}} \cdot \frac{L(\frac{1}{2}, \pi_{f_1} \times \pi_{f_{2,k}} \times \pi_{f_{3,k}})}{c_p^{-1}\Omega_{f_1}^b \Omega_{f_{2,k}}^b \Omega_{f_{3,k}}^b},$$

where  $\pi_{f_{i,k}}$  is the automorphic representation generated by  $\mathbf{f}_{i,k} = \mathbf{f}_i(\mathbf{u}^k - 1) \in \mathcal{S}_k(N_i p, 1; \overline{\mathbf{Q}})$ , and

$$\mathcal{E}^\dagger(s) = (-\alpha_1)\mathbf{a}_2(s)^{-1}\mathbf{a}_3(s)^{-1}p^{s+1}(1-\alpha_1 \cdot \mathbf{a}_2(s)^{-1}\mathbf{a}_3(s)^{-1}p^s)^2.$$

*(3) If  $\varepsilon_p(\mathbf{E}) = -1$ , then*

$$L_p^\dagger(0, s) = 0, \quad \frac{\partial L_p^\dagger}{\partial x}(0, 0) = (\ell_2 + \ell_3 - \ell_1)L_p^{\dagger\dagger}(0), \quad \text{ord}_{s=2}L_p(\mathbf{E}, s) \geq 3.$$

*(4) In Case (i),  $L_p^{\dagger\dagger}(0) = \frac{L(\mathbf{E}, 2)}{2^4\pi^5\Omega(\mathbf{E})}$ .*

**PROOF.** The construction of these improved  $L$ -functions are similar to that of  $L_{\mathbf{F},(a)}$  except that we need to replace the  $\Lambda_4$ -adic modular form  $\mathcal{G}_\chi^{(a)}$  in §6.5 with *improved* ones. To do so, we have to go back to §6.1 and modify the  $p$ -adic section  $f_{\mathcal{D},s,p}$  used in the construction of the Siegel Eisenstein series  $E_{\mathbf{A}}(g, f_{\mathcal{D},s,N}^{[k,\lambda]})$ . In the notation of Definition 2.5, for a datum  $\mathcal{D} = (\chi, \omega_1, \omega_2, \omega_3)$  of characters of  $\mathbf{Z}_p^\times$  and a Bruhat-Schwartz function  $\varphi_3 \in \mathcal{S}(\mathbf{Q}_p)$ , we modify the definition of Bruhat-Schwartz functions in (2.8) by

$$\Phi_{\mathcal{D}}(\varphi_1) \left( \begin{pmatrix} u_1 & x_3 & x_2 \\ x_3 & u_2 & x_1 \\ x_2 & x_1 & u_3 \end{pmatrix} \right) = \prod_{i=1}^3 \phi_i(u_i)\varphi_i(x_i),$$

where

$$\phi_1 = \phi_2 = \phi_3 = \widehat{\mathbb{I}}_{p\mathbf{Z}_p}, \quad \varphi_2 = \varphi_3 = \mathbb{I}_{\mathbf{Z}_p}.$$

Define the modified Bruhat-Schwartz functions by

$$\Phi_{\mathcal{D}}^{\dagger} = \Phi_{\mathcal{D}}(\widehat{\varphi}_{\chi\omega_1}), \quad \Phi_{\mathcal{D}}^{\dagger\dagger} = \Phi_{\mathcal{D}}(\mathbb{I}_{\mathbf{Z}_p}).$$

Following (2.5), we define the modified  $p$ -adic section  $f_{\mathcal{D},s}^{\bullet} := f_{\Phi_{\mathcal{D}}^{\bullet}}(\chi\widehat{\omega}\alpha_{\mathbf{Q}_p}^s)$  for  $\bullet \in \{\dagger, \dagger\dagger\}$ . Then the local degenerate Whittaker functions for these modified  $p$ -adic sections are given by

$$\mathcal{W}_B(f_{\mathcal{D},s}^{\dagger}) = (\chi\omega_1)(2b_{23})\mathbb{I}_{\Xi_p^{\dagger}}(B), \quad \mathcal{W}_B(f_{\mathcal{D},s}^{\dagger\dagger}) = \mathbb{I}_{\Xi_p^{\dagger\dagger}}(B),$$

for  $B = (b_{ij}) \in \text{Sym}_3(\mathbf{Q}_p)$ , where

$$\Xi_p^{\dagger} := \{(b_{ij}) \in \text{Sym}_3(\mathbf{Z}_p) \mid b_{11}, b_{22}, b_{33} \in p\mathbf{Z}_p\}, \quad \Xi_p^{\dagger\dagger} := \{(b_{ij}) \in \Xi_p^{\dagger} \mid 2b_{23} \in \mathbf{Z}_p^{\times}\}.$$

With the preparation above we define the power series

$$\begin{aligned} \mathcal{G}^{\dagger}(T, X) &= \sum_{B=(b_{ij}) \in T_3^+ \cap \Xi_p^{\dagger}} \langle 2b_{23} \rangle_T \langle 2b_{23} \rangle_X^{-1} \mathcal{F}_B^{(2)}(X, T, T, T) \cdot q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}} \in \mathbf{Z}_p[[T, X]][[q_1, q_2, q_3]], \\ \mathcal{G}^{\dagger\dagger}(T) &= \sum_{B \in T_3^+ \cap \Xi_p^{\dagger\dagger}} \mathcal{F}_B^{(2)}(T, T, T, T) \cdot q_1^{b_{11}} q_2^{b_{22}} q_3^{b_{33}} \in \mathbf{Z}_p[[T]][[q_1, q_2, q_3]]. \end{aligned}$$

Notation is as in §6.1. For arithmetic points  $(Q, P)$  with  $k_Q = 2$  we have

$$\mathcal{G}^{\dagger}(Q, P) = e_{\text{ord}} \mathbf{E}_{\mathcal{D}^{\dagger}, N}^{[k_P, r, \lambda]}(\tau, f_{\mathcal{D}^{\dagger}, s, N}^{\dagger})|_{s=0}, \quad \mathcal{G}^{\dagger\dagger}(P) = e_{\text{ord}} \mathbf{E}_{\mathcal{D}^{\dagger\dagger}, N}^{[k_P, r, \lambda]}(\tau, f_{\mathcal{D}^{\dagger\dagger}, s, N}^{\dagger\dagger})|_{s=0}$$

with  $\lambda = (0, 0, 0)$  and  $r = \frac{k_P}{2} - 1$ , where we have written

$$\begin{aligned} \mathcal{D}^{\dagger} &:= (\epsilon_P \omega^{2-k_P}, \epsilon_Q^{-1} \omega^{k_Q-2}, \epsilon_P^{-1} \omega^{k_P-2}, \epsilon_P^{-1} \omega^{k_P-2}), \\ \mathcal{D}^{\dagger\dagger} &:= (\epsilon_P \omega^{2-k_P}, \epsilon_P^{-1} \omega^{k_P-2}, \epsilon_P^{-1} \omega^{k_P-2}, \epsilon_P^{-1} \omega^{k_P-2}), \\ f_{\mathcal{D}^{\bullet}, s, N}^{\bullet} &:= f_{s, \infty}^{[k_P, \lambda]} \otimes f_{\mathcal{D}^{\bullet}, s}^{\bullet} \otimes f_{s, N} \otimes \ell_{Np}^0 f_{s, \ell}^0. \end{aligned}$$

As in Proposition 6.8 we can show that

$$\begin{aligned} \mathcal{G}^{\dagger}(T, X) &\in \mathbf{S}^{\text{ord}}(N, \omega^2, \mathbf{Z}_p[[X]]) \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{S}^{\text{ord}}(N, \omega^2, \mathbf{Z}_p[[T]]) \otimes_{\mathbf{Z}_p[[T]]} \mathbf{S}^{\text{ord}}(N, \omega^2, \mathbf{Z}_p[[T]]); \\ \mathcal{G}^{\dagger\dagger}(T) &\in \mathbf{S}^{\text{ord}}(N, \omega^2, \mathbf{Z}_p[[T]]) \otimes_{\mathbf{Z}_p[[T]]} \mathbf{S}^{\text{ord}}(N, \omega^2, \mathbf{Z}_p[[T]]) \otimes_{\mathbf{Z}_p[[T]]} \mathbf{S}^{\text{ord}}(N, \omega^2, \mathbf{Z}_p[[T]]). \end{aligned}$$

Choose an element  $H_i$  in the congruence ideal of  $\mathbf{f}_i$  with  $H_i(\mathbf{u}^2 - 1) \neq 0$ . We define the improved  $p$ -adic  $L$ -functions  $L_{\mathbf{F}, (2)}^{\dagger}(X, T)$  and  $L_{\mathbf{F}, (2)}^{\dagger\dagger}(T)$  as the first Fourier coefficients of

$$\begin{aligned} \mathbf{1}_{\mathbf{f}_1} \otimes \mathbf{1}_{\mathbf{f}_2} \otimes \mathbf{1}_{\mathbf{f}_3} (\text{Tr}_{N/N_1} \otimes \text{Tr}_{N/N_2} \otimes \text{Tr}_{N/N_3}(\mathcal{G}^{\dagger})) &\in \mathbf{Z}_p[[X, T]][\frac{1}{H^{\dagger}}]; \\ \mathbf{1}_{\mathbf{f}_1} \otimes \mathbf{1}_{\mathbf{f}_2} \otimes \mathbf{1}_{\mathbf{f}_3} (\text{Tr}_{N/N_1} \otimes \text{Tr}_{N/N_2} \otimes \text{Tr}_{N/N_3}(\mathcal{G}^{\dagger\dagger})) &\in \mathbf{Z}_p[[T]][\frac{1}{H^{\dagger\dagger}}] \end{aligned}$$

respectively, where  $H^{\dagger} = t_1 H_1(X) t_2 H_2 t_3 H_3(T)$  and  $H^{\dagger\dagger} = t_1 H_1 t_2 H_2 t_3 H_3(T)$ . Define

$$L_p^{\dagger}(x, s) := c_p \cdot \langle N^{-N^4} \rangle^{\frac{-s}{4}} L_{\mathbf{F}, (2)}^{\dagger}(\mathbf{u}^{x+2} - 1, \mathbf{u}^{s+2} - 1), \quad L_p^{\dagger\dagger}(s) := c_p \cdot \langle N^{-N^4} \rangle^{\frac{-s}{4}} L_{\mathbf{F}, (2)}^{\dagger\dagger}(\mathbf{u}^{s+2} - 1).$$

In view of the proof of Lemma 7.1, to prove the interpolation formulae for  $L_p^{\dagger}(x, s)$  and  $L_p^{\dagger\dagger}(s)$ , we need to compute the quantity  $Z_p^*(f_{\mathcal{D}, s}^{\bullet})$  defined in (2.13) attached to our modified  $p$ -adic sections  $f_{\mathcal{D}, s}^{\bullet}$  as well as a subrepresentation  $\pi_i$  of the induced representation  $I(\mu_i, \nu_i)$  of  $\text{GL}_2(\mathbf{Q}_p)$  with  $\mu_i$  unramified for  $i = 1, 2, 3$ . Applying the computation in Proposition 2.3, we find that whenever  $\chi\omega_2$  and  $\chi\omega_3$  are unramified,

$$Z_p^*(f_{\mathcal{D}, s}^{\dagger}) = Z_p^*(f_{\mathcal{D}, s}) \prod_{i=2,3} L\left(\frac{1}{2} - s, \chi^{-1} \mu_1^{-1} \mu_i^{-1} \nu_{5-i}^{-1}\right)$$

and that when  $\chi\omega_i$  are unramified for  $i = 1, 2, 3$ ,

$$Z_p^*(f_{\mathcal{D}, s}^{\dagger\dagger}) = Z_p^*(f_{\mathcal{D}, s}^{\dagger}) L\left(\frac{1}{2} - s, \chi^{-1} \nu_1^{-1} \mu_2^{-1} \mu_3^{-1}\right).$$

From the proof of Theorem 7.6 we can deduce the interpolation formulae for the improved  $L$ -functions. The formula for  $\mathcal{E}^+(s)$  follows from that for  $Z_p^*(f_{\mathcal{D},s})$  proved in Proposition 2.6 and Remark 3.5.

Whenever  $k > 2$ , the central sign for  $L(s, \pi_{f_1} \times \pi_{f_{2,k}} \times \pi_{f_{3,k}})$  is  $\varepsilon_p(\mathbf{E})$ . Therefore if  $\varepsilon_p(\mathbf{E}) = -1$ , then  $L_p^\dagger(0, s) = 0$  by (2), which implies that  $\frac{\partial L_p^\dagger}{\partial x}(0, 0) = \lim_{s \rightarrow 0} \frac{L_p^\dagger(s, s)}{s}$ . The second equality of (1) gives the expression of  $\lim_{s \rightarrow 0} \frac{L_p^\dagger(s, s)}{s}$ . We write

$$L_p(x, y, z, s) = \sum_{j=0}^{\infty} A_j(x, y, z) \left( s - \frac{x+y+z}{2} \right)^j.$$

If  $i \leq r := \text{ord}_{s=2} L_p(\mathbf{E}, s)$ , then

$$(8.2) \quad r = \min\{j \mid A_j(0, 0, 0) \neq 0\}, \quad \lim_{s \rightarrow 2} \frac{L_p(\mathbf{E}, s)}{(s-2)^i} = A_i(0, 0, 0).$$

Letting  $y = z = s = 0$ , we see by (1) that the power series

$$\sum_{j=0}^{\infty} A_j(x, 0, 0) \left( -\frac{x}{2} \right)^j = (1 - \alpha_1 \mathbf{a}_1(x)^{-1})^2 L_p^\dagger(x, 0)$$

has at least a double zero at  $x = 0$ . If  $\varepsilon_p(\mathbf{E}) = -1$ , then since  $A_{2n}(x, y, z) = 0$  for all non-negative integers  $n$  by the functional equation (8.1), we get  $A_1(0, 0, 0) = 0$  and  $r \geq 3$ .  $\square$

**8.4. The proof of Theorem 8.4(1).** We discuss Case (i). Then  $\varepsilon_p(\mathbf{E}) = -\varepsilon(\mathbf{E})$  by Remark 8.1. First suppose that  $\varepsilon(\mathbf{E}) = 1$ . The functional equation (8.1) allows us to write

$$L_p(x, y, z, s) = A_1(x, y, z) \left( s - \frac{x+y+z}{2} \right) + A_3(x, y, z) \left( s - \frac{x+y+z}{2} \right)^3 + \dots$$

The proof of Lemma 8.5(3) gives  $A_1(0, 0, 0) = 0$ . From (8.2) and Lemma 8.5(4) the formula boils down to

$$A_3(0, 0, 0) = -8\ell_1 \ell_2 \ell_3 L_p^{\dagger\dagger}(0).$$

If we denote the degree two term of  $A_1(x, y, z)$  by  $ax^2 + by^2 + cz^2 + dxy + eyz + fxz$ , then the degree three term of  $L_p(x, s, s, s)$  is given by

$$L^{(3)}(x, s) = \{ax^2 + (b+c+e)s^2 + (d+f)xs\}(-x/2) + A_3(0, 0, 0)(-x/2)^3.$$

On the other hand, from Lemma 8.5(1), (3) we find that

$$\begin{aligned} L^{(3)}(x, s) &= (\ell_1 x + (\ell_3 - \ell_2)s) \cdot (\ell_1 x + (\ell_2 - \ell_3)s)x \cdot \lim_{x \rightarrow 0} x^{-1} L_p^\dagger(x, 0) \\ &= (\ell_1^2 x^2 - (\ell_2 - \ell_3)^2 s^2)x \cdot (\ell_2 + \ell_3 - \ell_1) L_p^{\dagger\dagger}(0). \end{aligned}$$

Comparing the coefficients of  $x^2 s$ ,  $x s^2$  and  $x^3$ , we obtain the relations

$$d + f = 0, \quad b + c + e = 2(\ell_2 - \ell_3)^2(\ell_2 + \ell_3 - \ell_1) L_p^{\dagger\dagger}(0), \quad 4a + A_3(0, 0, 0) = -8\ell_1^2(\ell_2 + \ell_3 - \ell_1) L_p^{\dagger\dagger}(0).$$

By symmetry we get

$$\begin{aligned} d + e &= 0, & e + f &= 0; \\ a + c + f &= 2(\ell_1 - \ell_3)^2(\ell_1 + \ell_3 - \ell_2) L_p^{\dagger\dagger}(0), & a + b + d &= 2(\ell_1 - \ell_2)^2(\ell_1 + \ell_2 - \ell_3) L_p^{\dagger\dagger}(0). \end{aligned}$$

From these equations we conclude that  $d = e = f = 0$  and

$$\begin{aligned} a &= \{(\ell_1 - \ell_2)^2(\ell_1 + \ell_2 - \ell_3) + (\ell_1 - \ell_3)^2(\ell_1 + \ell_3 - \ell_2) - (\ell_2 - \ell_3)^2(\ell_2 + \ell_3 - \ell_1)\} L_p^{\dagger\dagger}(0), \\ A_3(0, 0, 0) &= -8\ell_1^2(\ell_2 + \ell_3 - \ell_1) L_p^{\dagger\dagger}(0) - 4a = -8\ell_1 \ell_2 \ell_3 L_p^{\dagger\dagger}(0). \end{aligned}$$

Next assume that  $\varepsilon(\mathbf{E}) = -1$ . Then  $\varepsilon_p(\mathbf{E}) = 1$ . By (8.1) and Lemma 8.5(1)

$$\sum_{n=0}^{\infty} A_{2n}(x, s, s) \left( \frac{s}{2} \right)^{2n} = \left( 1 - \frac{\mathbf{a}_2(s)}{\mathbf{a}_1(x)\mathbf{a}_3(s)} \right) \left( 1 - \frac{\mathbf{a}_3(s)}{\mathbf{a}_1(x)\mathbf{a}_2(s)} \right) L_p^\dagger(x, s).$$

Since  $L_p^\dagger(0, 0) = 0$ , every term in the right hand side has degree at least three. In particular, the constant term  $A_0(0, 0, 0)$  of the left hand side is zero. If we denote the degree two term of  $A_0(x, y, z)$  by  $\alpha x^2 + \beta y^2 + \gamma z^2 + \xi xy + \eta yz + \zeta xz$ , then the degree two term of the left hand side is

$$\alpha x^2 + (\beta + \gamma + \eta)s^2 + (\xi + \zeta)xs + A_2(0, 0, 0)(x/2)^2.$$

It is zero, and so by symmetry we get

$$\begin{aligned} A_2(0, 0, 0) &= -4\alpha, & \beta + \gamma + \eta &= 0, & \xi + \zeta &= 0; \\ A_2(0, 0, 0) &= -4\beta, & \alpha + \gamma + \zeta &= 0, & \xi + \eta &= 0; \\ A_2(0, 0, 0) &= -4\gamma, & \alpha + \beta + \xi &= 0, & \eta + \zeta &= 0. \end{aligned}$$

We arrive at  $\xi = \eta = \zeta = \alpha = \beta = \gamma = A_2(0, 0, 0) = 0$ . Hence  $\text{ord}_{s=2}L_p(\mathbf{E}, s) \geq 4$ .

**8.5. The proof of Theorem 8.4(2).** We discuss Case (ii). Then  $\varepsilon_p(\mathbf{E}) = \varepsilon(\mathbf{E})$  by Remark 8.1. If  $\varepsilon(\mathbf{E}) = -1$ , then  $\text{ord}_{s=2}L_p(\mathbf{E}, s) \geq 3$  by Lemma 8.5(3), and both sides of the declared identity are zero. We will consider the case  $\varepsilon(\mathbf{E}) = 1$ , i.e.  $\Sigma^-$  has odd cardinality. Unlike Case (i) we cannot apply Lemma 8.5(3). Our proof relies on the three-variable  $p$ -adic triple product  $L$ -function in the balanced case constructed in [Hsi19].

Let  $D$  be the definite quaternion algebra over  $\mathbf{Q}$  of discriminant  $N^-$  and  $\mathbf{S}^D(N, \Lambda)$  the space of  $\Lambda$ -adic modular forms on  $D^\times$  defined in [Hsi19, Definition 4.1]. Let  $\mathbf{f}_i^D \in \mathbf{S}^D(N, \Lambda)[t_i^{-1}]$  be a Jacquet-Langlands lift of  $\mathbf{f}_i$  in the sense of [Hsi19, §4.5]. Since we do not assume Hypothesis (CR,  $\Sigma^-$ ) of [Hsi19, §1.4], we cannot choose  $\mathbf{f}_i^D$  to be a primitive Jacquet-Langlands lift as in [Hsi19, Theorem 4.5]. Nonetheless,  $\mathbf{f}_i^D$  can be chosen so that  $\mathbf{f}_i^D(\mathbf{u}^2 - 1)$  is a non-zero Jacquet-Langlands lift of  $f_i$ . Replacing the triple  $\mathbf{F}^D = (\mathbf{f}_1^D, \mathbf{f}_2^D, \mathbf{f}_3^D)$  with the well-chosen test vectors in [Hsi19, Definition 4.8], we can associate to  $\mathbf{F}^D$  the three-variable *theta element*  $\Theta_{\mathbf{F}^D}(X_1, X_2, X_3)$  in *loc.cit.* Define an analytic function on  $\mathcal{U}^3 \subset \mathbf{Z}_p^3$  by

$$\Theta(x, y, z) = \Theta_{\mathbf{F}^D}(\mathbf{u}^{x+2} - 1, \mathbf{u}^{y+2} - 1, \mathbf{u}^{z+2} - 1).$$

By the interpolation formula for  $\Theta_{\mathbf{F}^D}$  in [Hsi19, Theorem 7.1] (see Remark 7.8), we can find an analytic function  $H(x, y, z)$  with  $H(0, 0, 0) \neq 0$  such that

$$H(x, y, z) \cdot \Theta(x, y, z)^2 = L_p\left(x, y, z, \frac{x+y+z}{2}\right).$$

To proceed, we introduce two-variable *improved* theta elements.

**Lemma 8.6** (Improved theta elements). *There exist analytic functions  $\Theta_2^\dagger(x, z)$ ,  $\Theta_3^\dagger(x, y)$  such that*

$$\begin{aligned} \Theta_2^\dagger(0, 0) &= -\Theta_3^\dagger(0, 0), \\ \Theta(x, x+z, z) &= \left(1 - \frac{\mathbf{a}_2(x+z)}{\mathbf{a}_1(x)\mathbf{a}_3(z)}\right) \Theta_2^\dagger(x, z), & \Theta(x, y, x+y) &= \left(1 - \frac{\mathbf{a}_3(x+y)}{\mathbf{a}_1(x)\mathbf{a}_2(y)}\right) \Theta_3^\dagger(x, y). \end{aligned}$$

**PROOF.** The idea of the proof is similar to [Hsi19, Proposition 8.3]. We give a sketch of the proof here. For every integer  $n$ , let  $R_n$  be the Eichler order of level  $p^n N/N^-$  in  $D$  and let  $X_0(p^n N) = D^\times \backslash \widehat{D}^\times / \widehat{R}_n^\times$ , where  $\widehat{D} = D \otimes \widehat{\mathbf{Q}}$  and  $\widehat{R}_n = R_n \otimes \widehat{\mathbf{Z}}$ . Through an isomorphism  $R_0 \otimes \mathbf{Z}_p \simeq \mathbf{M}_2(\mathbf{Z}_p)$  we define

$$U_1(p^n) := \left\{ g \in \widehat{R}_n \mid g_p \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{p^n} \right\}.$$

Recall that  $\mathbf{a}_i(Q) = \mathbf{a}(p, \mathbf{f}_{i,Q})$  and that  $\varpi_p \in \widehat{\mathbf{Q}}^\times$  is the element with  $\varpi_{p,p} = p$  and  $\varpi_{p,\ell} = 1$  for  $\ell \neq p$ . For all but finitely many arithmetic points  $Q$  with  $k_Q = 2$ , the specialization  $\mathbf{f}_{i,Q}^D : D^\times \backslash \widehat{D}^\times / U_1(p^n) \rightarrow \mathbf{C}_p$  is a  $p$ -stabilized form on  $\widehat{D}^\times$  with the same Hecke eigenvalues with  $\mathbf{f}_{i,Q}$  and the central character  $\epsilon_Q^{-1} : \mathbf{Q}^\times \backslash \widehat{\mathbf{Q}}^\times / (1 + p^n \widehat{\mathbf{Z}})^\times \rightarrow \mu_{p^\infty}$  for any sufficiently large  $n$ . In particular,  $\mathbf{f}_{i,Q}^D$  is a  $\mathbf{U}_p$ -eigenform with eigenvalue  $\mathbf{a}_i(Q)$ . Namely,

$$(8.3) \quad \mathbf{U}_p \mathbf{f}_{i,Q}^D(g) := \sum_{b \in \mathbf{Z}_p/p^n \mathbf{Z}_p} \mathbf{f}_{i,Q}^D\left(g \begin{pmatrix} \varpi_p^n & b \\ 0 & 1 \end{pmatrix}\right) = \mathbf{a}_i(Q)^n \mathbf{f}_{i,Q}^D(g), \quad g \in \widehat{D}^\times.$$

In what follows, we shall write  $(\mathbf{f}^D, \mathbf{g}^D, \mathbf{h}^D) = (\mathbf{f}_1^D, \mathbf{f}_2^D, \mathbf{f}_3^D)$ . Let  $N_D : D \rightarrow \mathbf{Q}$  be the reduced norm. Put  $\tau_{p^n} = \begin{pmatrix} 0 & 1 \\ -\varpi_p^n & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_p) \subset \widehat{D}^\times$ . By definition,

$$(8.4) \quad \Theta(Q_1, Q_2, Q_3) = \mathbf{a}_1(Q_1)^{-n} \mathbf{a}_2(Q_2)^{-n} \mathbf{a}_3(Q_3)^{-n} \\ \sum_{[a] \in X_0(p^n N)} \sum_{\substack{b \in \mathbf{Z}_p/p^n \mathbf{Z}_p \\ c \in (\mathbf{Z}_p/p^n \mathbf{Z}_p)^\times}} \mathbf{f}_{Q_1}^D \left( a \begin{pmatrix} \varpi_p^n & b \\ 0 & 1 \end{pmatrix} \right) \mathbf{g}_{Q_2}^D \left( a \begin{pmatrix} \varpi_p^n & b+c \\ 0 & 1 \end{pmatrix} \right) \mathbf{h}_{Q_3}^D(a\tau_{p^n}) \epsilon_{Q_1 Q_2 Q_3}^{\frac{1}{2}}(c) \epsilon_{Q_1 Q_2 Q_3}^{\frac{1}{2}}(N_D(a)).$$

We replace the twisted diagonal cycle  $\Delta_n$  in [Hsi19, Definition 4.6] by the *improved* diagonal cycle

$$\Delta_n^\ddagger := \sum_{[a] \in X_0(Np^n)} \sum_{b \in \mathbf{Z}_p/p^n \mathbf{Z}_p} \left[ \left( a \begin{pmatrix} \varpi_p^n & b \\ 0 & 1 \end{pmatrix}, a\tau_{p^n}, a \right) \right].$$

We can define the regularized improved diagonal cycle by

$$\Delta_\infty^\ddagger := \varprojlim_{n \rightarrow \infty} (\mathbf{U}_p^{-n} \otimes \mathbf{U}_p^{-n} \otimes 1) e_E(\Delta_n^\ddagger),$$

and the improved theta element

$$\Theta_2^\ddagger(X_1, X_3) := (\mathbf{F}^D)^*(\Delta_\infty^\ddagger)(X_1, (1+X_1)(1+X_3)-1, X_3) \in \mathbf{Z}_p[[X_1, X_3]][t^{-1}].$$

for  $t = t_1 \cdot t_2 \cdot ((1+X_1)(1+X_3)-1) \cdot t_3$ . Put  $\Theta_2^\ddagger(x, z) := \Theta_2^\ddagger(\mathbf{u}^{x+2}-1, \mathbf{u}^{z+2}-1)$  for  $(x, z) \in \mathcal{U}^2$ . By definition and (8.3), for all but finitely many arithmetic points  $(Q_1, Q_3)$  with  $k_{Q_1} = k_{Q_3} = 2$

$$\Theta_2^\ddagger(Q_1, Q_3) = \mathbf{a}_2(Q_1 Q_3)^{-n} \sum_{[a] \in X_0(Np^n)} \mathbf{f}_{Q_1}^D(a) \mathbf{g}_{Q_1 Q_3}^D(a\tau_{p^n}) \mathbf{h}_{Q_3}^D(a) \epsilon_{Q_1 Q_3}(N_D(a)).$$

The above expression holds for any  $n$  such that  $p^n$  is bigger than the conductors of  $\epsilon_{Q_1}$  and  $\epsilon_{Q_2}$ . Likewise we can define  $\Theta_3^\ddagger \in \mathbf{Z}_p[[X_1, X_2]]$  and  $\Theta_3^\ddagger(x, y)$  with the interpolation property:

$$\Theta_3^\ddagger(Q_1, Q_2) = \mathbf{a}_3(Q_1 Q_2)^{-n} \sum_{[a] \in X_0(Np^n)} \mathbf{f}_{Q_1}^D(a\tau_{p^n}) \mathbf{g}_{Q_2}^D(a) \mathbf{h}_{Q_1 Q_2}^D(a) \epsilon_{Q_1 Q_2}(N_D(a)).$$

To see the first relation, we note that

$$\Theta_2^\ddagger(0, 0) = \alpha_2^{-1} \sum_{[a] \in X_0(Np)} \mathbf{f}_0^D(a) \mathbf{g}_0^D(a\tau_p) \mathbf{h}_0^D(a), \quad \Theta_3^\ddagger(0, 0) = \alpha_3^{-1} \sum_{[a] \in X_0(Np)} \mathbf{f}_0^D(a) \mathbf{g}_0^D(a) \mathbf{h}_0^D(a\tau_p).$$

Since  $\mathbf{f}_0^D$  is a newform that is special at  $p$ ,  $\mathbf{f}_0^D(x\tau_p) = (-\alpha_1) \mathbf{f}_0^D(x)$ , and hence  $\Theta_2^\ddagger(0, 0) = -\Theta_3^\ddagger(0, 0)$ .

To prove the last relation, it suffices to verify the following equation

$$(8.5) \quad \Theta(Q_1, Q_1 Q_3, Q_3) = \left( 1 - \frac{\mathbf{a}_2(Q_1 Q_3)}{\mathbf{a}_1(Q_1) \mathbf{a}_3(Q_3)} \right) \Theta_2^\ddagger(Q_1, Q_3)$$

for all but finitely many arithmetic points  $(Q_1, Q_3)$  with  $k_{Q_1} = k_{Q_3} = 2$ . The formula for  $\Theta_3^\ddagger$  can be done by a similar computation, so we leave it to the reader. Let  $n$  be a sufficiently large integer. From (8.4), we get

$$\mathbf{a}_1(Q_1)^n \mathbf{a}_2(Q_1 Q_3)^n \mathbf{a}_3(Q_3)^n p^{-n} \mathrm{vol}(\widehat{R}_n^\times) \Theta(Q_1, Q_1 Q_3, Q_3) \\ = \int_{D^\times \setminus \widehat{D}^\times} d^\times a \sum_{c \in (\mathbf{Z}_p/p^n \mathbf{Z}_p)^\times} \mathbf{f}_{Q_1}^D \left( a \begin{pmatrix} 1 & -c\varpi_p^{-n} \\ 0 & 1 \end{pmatrix} \right) \mathbf{g}_{Q_1 Q_3}^D(a) \mathbf{h}_{Q_3}^D \left( a\tau_{p^n} \begin{pmatrix} 1 & 0 \\ 0 & \varpi_p^{-n} \end{pmatrix} \right) \epsilon_{Q_1}(c) \epsilon_{Q_1 Q_3}(N_D(a)) \\ = \int_{D^\times \setminus \widehat{D}^\times} d^\times a \sum_{c \in (\mathbf{Z}_p/p^n \mathbf{Z}_p)^\times} \mathbf{f}_{Q_1}^D \left( a\tau_{p^n} \begin{pmatrix} 1 & -\varpi_p^{-n} \\ 0 & c^{-1} \end{pmatrix} \right) \mathbf{g}_{Q_1 Q_3}^D(a\tau_{p^n}) \mathbf{h}_{Q_3}^D \left( a \begin{pmatrix} 1 & 0 \\ 0 & \varpi_p^{-n} \end{pmatrix} \right) \epsilon_{Q_1 Q_3}(N_D(a))$$

by change of variables. From the equations  $\tau_{p^n} \begin{pmatrix} 1 & -\varpi_p^{-n} \\ 0 & c^{-1} \end{pmatrix} = \begin{pmatrix} \varpi_p^n & c^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c^{-1} & 0 \\ -\varpi_p^n & 1 \end{pmatrix}$ ,  $\epsilon_{Q_1}(\varpi_p) = \epsilon_{Q_3}(\varpi_p) = 1$  and (8.3), the last integral equals

$$\begin{aligned} & \int_{D^\times \setminus \widehat{D}^\times} d^\times a \sum_{c \in (\mathbf{Z}_p/p^n \mathbf{Z}_p)^\times} \mathbf{f}_{Q_1}^D \left( a \begin{pmatrix} \varpi_p^n & c \\ 0 & 1 \end{pmatrix} \right) \mathbf{g}_{Q_1 Q_3}^D(a\tau_{p^n}) \mathbf{h}_{Q_3}^D \left( a \begin{pmatrix} \varpi_p^n & 0 \\ 0 & 1 \end{pmatrix} \right) \epsilon_{Q_1 Q_3}(\mathbf{N}_D(a)) \\ &= \int_{D^\times \setminus \widehat{D}^\times} d^\times a \mathbf{a}_1(Q_1)^n \cdot \left\{ \mathbf{f}_{Q_1}^D(a) - \mathbf{a}_1(Q_1)^{-1} \mathbf{f}_{Q_1}^D \left( a \begin{pmatrix} \varpi_p & 0 \\ 0 & 1 \end{pmatrix} \right) \right\} \mathbf{g}_{Q_1 Q_3}^D(a\tau_{p^n}) \mathbf{h}_{Q_3}^D \left( a \begin{pmatrix} \varpi_p^n & 0 \\ 0 & 1 \end{pmatrix} \right) \epsilon_{Q_1 Q_3}(\mathbf{N}_D(a)) \\ &= \mathbf{a}_1(Q_1)^n \int_{D^\times \setminus \widehat{D}^\times} d^\times a \mathbf{f}_{Q_1}^D(a) \mathbf{g}_{Q_1 Q_3}^D(a\tau_{p^n}) p^{-n} \sum_{b \in \mathbf{Z}_p/p^n \mathbf{Z}_p} \mathbf{h}_{Q_3}^D \left( a \begin{pmatrix} \varpi_p^n & b \\ 0 & 1 \end{pmatrix} \right) \epsilon_{Q_1 Q_3}(\mathbf{N}_D(a)) \\ &\quad - \mathbf{a}_1(Q_1)^{n-1} \int_{D^\times \setminus \widehat{D}^\times} d^\times a \mathbf{f}_{Q_1}^D(a) \mathbf{g}_{Q_1 Q_3}^D(a\tau_{p^{n+1}}) p^{-(n-1)} \sum_{b \in \mathbf{Z}_p/p^{n-1} \mathbf{Z}_p} \mathbf{h}_{Q_3}^D \left( a \begin{pmatrix} \varpi_p^{n-1} & b \\ 0 & 1 \end{pmatrix} \right) \epsilon_{Q_1 Q_3}(\mathbf{N}_D(a)) \\ &= \{ (\mathbf{a}_1(Q_1) \mathbf{a}_3(Q_3) \mathbf{a}_2(Q_1 Q_3)/p)^n \text{vol}(\widehat{R}_n^\times) - (\mathbf{a}_1(Q_1) \mathbf{a}_3(Q_3)/p)^{n-1} \mathbf{a}_2(Q_1 Q_3)^{n+1} \text{vol}(\widehat{R}_{n+1}^\times) \} \Theta_2^\dagger(Q_1, Q_3) \\ &= \mathbf{a}_1(Q_1)^n \mathbf{a}_3(Q_3)^n \mathbf{a}_2(Q_1 Q_3)^n \left( 1 - \frac{\mathbf{a}_2(Q_1 Q_3)}{\mathbf{a}_1(Q_1) \mathbf{a}_3(Q_3)} \right) p^{-n} \text{vol}(\widehat{R}_n^\times) \Theta_2^\dagger(Q_1, Q_3). \end{aligned}$$

This verifies (8.5).  $\square$

Now we return to the proof of Theorem 8.4(2). Write  $\Theta_x$  for the partial derivative  $\frac{\partial \Theta}{\partial x}$ . Put

$$a = \Theta_x(0, 0, 0), \quad b = \Theta_y(0, 0, 0), \quad c = \Theta_z(0, 0, 0).$$

Taking derivatives  $\Theta(x, y, x+y)$  with respect to  $x$  and  $y$  at  $(0, 0)$  in Lemma 8.6, we have

$$a + c = (\ell_1 - \ell_3) \Theta_3^\dagger(0, 0), \quad b + c = (\ell_2 - \ell_3) \Theta_3^\dagger(0, 0).$$

Similarly, we have

$$a + b = (\ell_1 - \ell_2) \Theta_2^\dagger(0, 0) = (\ell_2 - \ell_1) \Theta_3^\dagger(0, 0).$$

These imply that

$$a = 0, \quad b = (\ell_2 - \ell_1) \Theta_3^\dagger(0, 0), \quad c = (\ell_1 - \ell_3) \Theta_3^\dagger(0, 0).$$

On the other hand, by the functional equation (8.1) we obtain the Taylor expansion

$$L_p(x, y, z, s) = H(x, y, z) \Theta(x, y, z)^2 + A_2(x, y, z) \cdot \left( s - \frac{x+y+z}{2} \right)^2 + \dots$$

By Lemma 8.5(1), we find

$$(1 - \alpha_1 \mathbf{a}_1(x)^{-1})^2 L_p^\dagger(x, 0) = H(x, 0, 0) \Theta(x, 0, 0)^2 + A_2(x, 0, 0) \cdot x^2/4.$$

From the vanishing of  $\Theta_x(0, 0, 0)$  we deduce that

$$A_2(0, 0, 0) = 4\ell_1^2 L_p^\dagger(0, 0).$$

Lemma 8.5(2) and (8.2) complete the proof of Theorem 8.4(2).

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